Solutions To Project 7

(1) $y - f(x_0) = f'(x_0)(x - x_0)$

(a)

$$f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x}$$

$$x_0 = 1 \Rightarrow f(x_0) = \ln(1) = 0$$

$$f'(x_0) = \frac{1}{1} = 1$$

So $y - 0 = 1(x-1)$

$$y = x - 1$$

(b)

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$$

$$x_0 = 1 \Rightarrow f(x_0) = \frac{1}{1} = 1$$

$$f'(x_0) = -\frac{1}{1^2} = -1$$

So $y - 1 = -1(x-1)$

$$y - 1 = -x + 1$$

$$y = -x + 2$$

(c)

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

$$x_0 = 1 \Rightarrow f(x_0) = \sqrt{1} = 1$$

$$f'(x_0) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

So $y - 1 = \frac{1}{2}(x-1)$

$$y - 1 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

(d)

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2$$

$$x_0 = 0 \Rightarrow f(x_0) = 0^3 = 0$$

$$f'(x_0) = 3(0)^2 = 0$$

So $y - 0 = 0(x - 0)$

$y = 0$; the X-axis

(2) $$f(x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)(\Delta x)^2 + \frac{1}{6} f'''(x_0)(\Delta x)^3$$

(a)

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

$$x_0 = 0 \Rightarrow f(x_0) = e^0 = 1$$

$$f'(x_0) = e^0 = 1$$

So $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

(b)

$$f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}$$

$$x_0 = 1 \Rightarrow f(x_0) = \ln(1) = 0$$

$$f'(x_0) = \frac{1}{1} = 1$$

$$f''(x_0) = -\frac{1}{1^2} = -1$$

$$f'''(x_0) = \frac{2}{1^3} = 2$$

So $\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

(c)

$$f(x) = xe^x \Rightarrow f'(x) = 1.e^x + xe^x$$
$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$$
$$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$$
$$x_0 = 0 \Rightarrow f(x_0) = 0$$
$$f'(x_0) = 1$$
$$f''(x_0) = 2$$
$$f'''(x_0) = 3$$
$$\text{So} \quad xe^x = x + x^2 + \frac{1}{2}x^3$$

(d)

$$f(x) = \sqrt{x} \Rightarrow f'(x_0) = \frac{1}{2\sqrt{x}}$$
$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}},\ f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$
$$x_0 = 1 \Rightarrow f(x_0) = 1,\ f'(x_0) = \frac{1}{2}$$
$$f''(x_0) = -\frac{1}{4},\ f'''(x_0) = \frac{3}{8}$$
$$\text{So} \quad \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

(3) $$\Delta W = \frac{dW}{dx}\Delta x + \frac{1}{2}\frac{d^2W}{dx^2}(\Delta x)^2 ; W_0 = 10$$

(a)

$$\Delta x = +.5 \Rightarrow \Delta W = -2(.5) + \frac{1}{2}(-1)(.5)^2 = -1.125$$
$$\text{So} \quad W_{new} = 10 + \Delta W = 10 - 1.125 = 8.875$$

The loss is $1.125 per share or $112.50 total

(b)

$$\Delta x = -1 \Rightarrow \Delta W = -2(-1) + \frac{1}{2}(-1)(-1)^2 = 1.5$$
$$\text{So} \quad W_{\text{new}} = 10 + \Delta W = 10 + 1.5 = 11.5$$

The profit is $1.50 per share or $150 total

(4)

$2^x + 3^x = 1{,}000{,}000$

x	12	13
2^x+3^x	535,537	1,602,515

Clearly the solution $\tilde{x}$ is between 12 and 13.

Guess Solution $x_0 = 12.5$

$g(x) = 2^x + 3^x - 10^6$

$g'(x) = \ln(2)2^x + \ln(3)3^x$

$g'(x) = .693(2^x) + 1.0986.(3^x)$

$x_j = x_{j-1} - \dfrac{g(x_{j-1})}{g'(x_{j-1})}$

j	x_j	$g(x_j)$	$g'(x_j)$	$g(x_j)/g'(x_j)$	$x_j - g(x_j)/g'(x_j)$
0	12.5	73,725	1.02×10^6	.07228	12.57228
1	12.57228	2646.9	1.10×10^6	.002406	12.56987
2	12.56987	1.7001	1.10×10^6	.0000015	12.56986

We accept $\tilde{x} = 12.56986$ A TI-82 guess 12.56986845

(5)

$90 = \dfrac{5}{1+R} + \dfrac{5}{(1+R)^2} + \dfrac{5}{(1+R)^3} + \dfrac{105}{(1+R)^4}; 0 < R$ Note: $\dfrac{1}{x} = 1 + R \Rightarrow R = \dfrac{1}{x} - 1$

Let $x = \dfrac{1}{1+R} \Rightarrow 105x^4 + 5x^3 + 5x^2 + 5x - 90 = 0$

$g(x) = 105x^4 + 5x^3 + 5x^2 + 5x - 90 \Rightarrow g'(x) = 420x^3 + 15x^2 + 10x + 5$

$x_j = x_{j-1} - \dfrac{g(x_j)}{g'(x_j)}$; $x_0 = .95$

j	x_j	$g(x_j)$	$g'(x_j)$	$g(x_j)/g'(x_j)$	$x_j - g(x_j)/g'(x_j)$
0	.95	9.0725	388.14	.02337	.92662
1	.92662	.31412	381.31	.00082	.92579
2	.92579	.01462	360.38	.00004	.92575

We accept $\tilde{x} = .92575 \Rightarrow \tilde{R} = \dfrac{1}{.92575} - 1 = .08$

The yield is 16% per annum compounded semiannually!

(6)

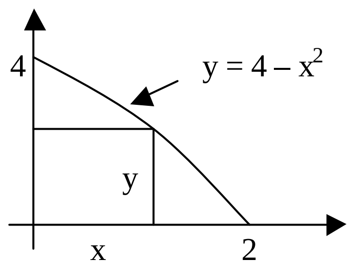
$$\left.\begin{array}{l} xe^x = 1{,}000 \\ xe^x - 10^3 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} g(x) = xe^x - 10^3 \\ g'(x) = e^x + xe^x \end{array}\right\} \Rightarrow x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

$$x_1 = 5 - \frac{(-257.934)}{890.479}$$

$$x_1 = 5 + .2896 = 5.29$$

A TI-82 guess 5.249 as solution!

(7)



$A = xy = x(4 - x^2) = 4x - x^3 = f(x)$

$f'(x) = 4 - 3x^2$

$f'(x) = 0 \Leftrightarrow x^2 = \frac{4}{3}$

$x = 1.155$

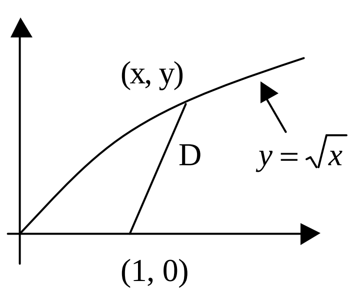
for clearly $0 < x$

x	0<x<1.155	1.155	1.155<x<
f'(x)	positive	0	negative
f(x)	increasing		decreasing

So $x_{max} = 1.155$

$y_{max} = 4 - (1.155)^2 = 2.667 \Rightarrow A_{max} = 3.08$ Sq. Units

(8)



$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$D = \sqrt{(x-1)^2 + y^2}$$
$$D = \sqrt{(x-1)^2 + x}$$
$$D = \sqrt{x^2 - x + 1} = f(x)$$
$$f'(x) = \frac{2x-1}{2\sqrt{x^2 - x + 1}}$$
$$f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$$

X	0≤X<½	½	½<X
f'(X)	Negative	0	Positive
f(X)	Decreasing		Increasing

The point is $(.5, .707)$
and $D_{\min} = .866$

(9)

$$y = f(x) = x^3 - 6x^2 + 9x + 6$$

x	-3	-2	-1	0	1	2	3	4
y	-102	-44	-10	6	10	8	6	10

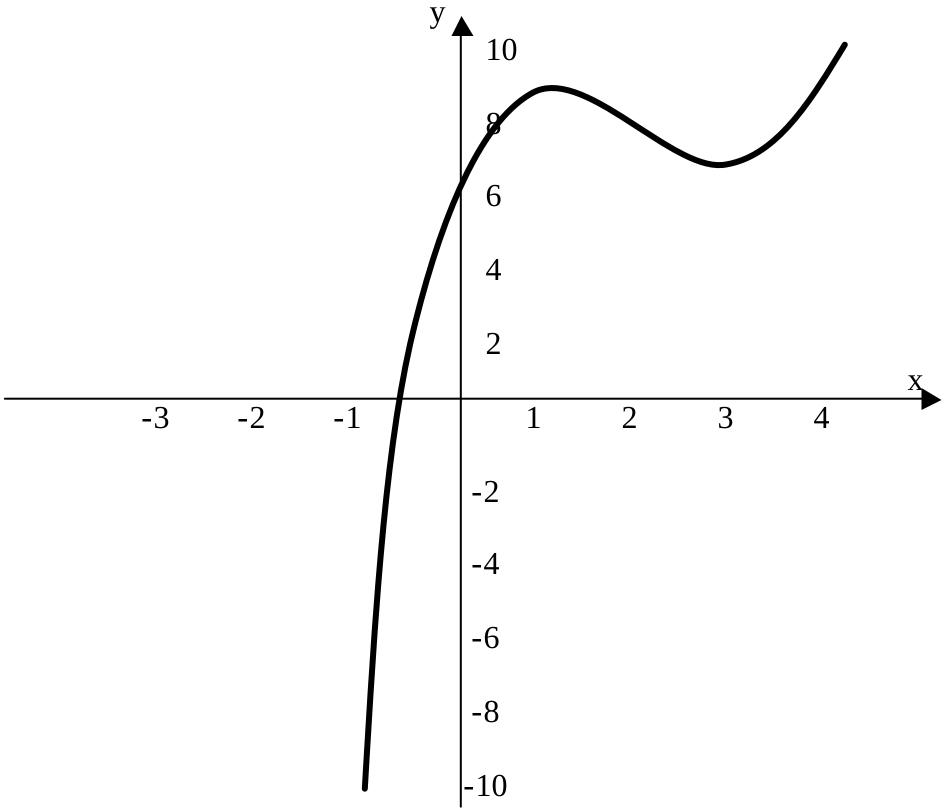
$$y' = 3x^2 - 12x + 9$$
$$y' = 3(x^2 - 4x + 3)$$
$$y' = 3(x-1)(x-3)$$
$y' = 0$ if and only if $x = 1$ or $x = 3$
$$y'' = 6x - 12 = 6(x-2)$$
$y'' = 0$ if and only if $x = 2$

x	x<1	1	1<x<3	3	3<x
y'	positive	0	negative	0	positive
y	increasing	maximum	decreasing	minimum	increasing

x = 1 is a local maximum
x = 3 is a local minimum

x	x<2	2	2<x
y''	negative	0	positive
y	frowning	inflex	smiling

x = 2 is an inflection point.



Note: one x-intercept, it is between –1 and 0

(10)

$$y = f(x) = x^4 - 6x^2$$

Note: f is symmetric about y-axis.

x	-3	-2	-1	0	1	2	3
y	27	-8	-5	0	-5	-8	27