Solution To Project 5

1.

x-y-z=1

2x-3y+z=10

x+y-2z=0

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & -3 & 1 & 10 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & -1 & 3 & 8 \\ 0 & 2 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -1R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & -3 & -8 \\ 0 & 2 & -1 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2\ ,\ R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 + 4R_3\ ,\ R_2 \to R_2 + 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So x =5, y =1, and z =3.

Let's check:

(5)- (1)- (3)=1

2(5)-3(1)+(3)=10

(5)+(1)-2(3)=0

2.

x- y+ z = 6

3x- y+ 11z = 6

2x+ y+ 4z = 8

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 3 & -1 & 11 & 6 \\ 2 & 1 & 4 & 8 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 2 & 8 & -12 \\ 0 & 3 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 4 & -6 \\ 0 & 3 & 2 & -4 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2\ ,\ R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 4 & -6 \\ 0 & 0 & -10 & 14 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{10}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 4 & -6 \\ 0 & 0 & 1 & -1.4 \end{array}\right] \xrightarrow{R_1 \to R_1 - 5R_3\ ,\ R_2 \to R_2 - 4R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -.4 \\ 0 & 0 & 1 & -1.4 \end{array}\right]$$

So x =7 , y = -.4, and z = -1.4 .

Let’s check:

(7)- (-.4)+ (-1.4)=6

3(7)- (-.4)+11(-1.4)=6

2(7)+ (-.4)+ 4(-1.4)=8

3.

x + 5y +3z =7

2x +11y-4z=6

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 7 \\ 2 & 11 & -4 & 6 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 7 \\ 0 & 1 & -10 & -8 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 53 & 47 \\ 0 & 1 & -10 & -8 \end{array}\right]$$

So x + 53z = 47 or x = -53z + 47
y-10z = - 8 y =10z – 8

These are an infinite number of solutions!

Some solutions are :

x	y	z
47,	-8,	0
-6,	2,	1
100,	-18,	-1

4.
x+y+z=6

3x+2y-z=4

3x+y+2z=11

a.

1	1	1	1	1	1	1	1	1	1	1	1
3	2	-1	3	2	-1	3	2	-1	3	2	-1
3	1	2	3	1	2	3	1	2	3	1	2

1 x 2 x2= 4

1 x(-1) x3=-3

1 x 3 x1= 3

4

3 x 2 x 1= 6

1 x(-1) x 1=-1

2 x 3 x 1= 6

11

So $\det\begin{bmatrix}1 & 1 & 1\\ 3 & 2-1\\ 3 & 1 & 2\end{bmatrix} = 4-11=-7=D$

b.

6	1	1	6	1	1	6	1	1	6	1	1
4	2	-1	4	2	-1	4	2	-1	4	2	-1
11	1	2	11	1	2	11	1	2	11	1	2

6 x 2 x 2 = 24

1 x(-1)x11 =-11

1 x 4 x 1 = 4

17

11 x 2 x1 =22

1 x(-1)x6 =-6

2 x 4 x1 = 8

24

So det $\begin{bmatrix} 6 & 1 & 1 \\ 4 & 2 & -1 \\ 11 & 1 & 2 \end{bmatrix} = 17 - 24 = -7 = D_X$

c.

1	6	1	1	6	1	1	6	1	1	6	1
3	4	-1	3	4	-1	3	4	-1	3	4	-1
3	11	2	3	11	2	3	11	2	3	11	2

1 x 4 x 2 = 8

6 x(-1)x 3 =-18

1 x 3 x11 = 33

____

23

3 x 4 x1 = 12

11 x(-1)x1 =-11

2 x 3 x6 = 36

____

37

So det $\begin{bmatrix} 1 & 6 & 1 \\ 3 & 4 & -1 \\ 3 & 11 & 2 \end{bmatrix} = 23 - 37 = -14 = D_Y$

d.

1	1	6	1	1	6	1	1	6	1	1	6
3	2	4	3	2	4	3	2	4	3	2	4
3	1	11	3	1	11	3	1	11	3	1	11

1 x 2 x 11 = 22	3 x 2 x6 = 36
1 x 4 x 3 = 12	1 x 4 x1 = 4
6 x 3 x 11 = 18	11x 3 x1 = 33
52	73

So det $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 2 & 4 \\ 3 & 1 & 11 \end{bmatrix} = 52 - 73 = -21 = D_Z$

Cramer's Rule says

$$x = \frac{D_x}{D} = \frac{-7}{-7} = 1$$

$$y = \frac{D_y}{D} = \frac{-14}{-7} = 2$$

$$z = \frac{D_z}{D} = \frac{-21}{-7} = 3$$

Let's check: (1)+(2)+(3) = 6

3(1)+2(2)-(3)= 4

3(1)+2+2(3) =11

5.

$$x^2 + y^2 = 4$$

$$-2x + y = -1$$

$$-2x + y = -1 \Rightarrow y = 2x - 1$$

$$But \quad x^2 + y^2 = 4 \quad so \quad x^2 + (2x-1)^2 = 4$$

$$and \quad x^2 + (4x^2 - 4x + 1) = 4$$

$$and \; 5x^2 - 4x + 1 = 4$$

$$thus \;\; 5x^2 - 4x - 3 = 0$$

This quadratic equation has solutions:

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-3)}}{2(5)} = \frac{4 \pm \sqrt{76}}{10} = \frac{4 \pm 8.72}{10}$$

These are two solutions to the quadratic equation:

$$x_1 = \frac{4 + 8.72}{10} = 1.27 \quad and \quad x_2 = \frac{4 - 8.72}{10} = -.47$$

$$But \;\; y = 2x + 1 \;\; so \;\; if \quad x_1 = 1.27 \quad then \quad y_1 = 2(1.27) - 1 = 1.54$$

$$and \;\; if \quad x_2 = -.47 \quad then \quad y_2 = 2(-.47) - 1 = -.94 - 1 = -1.94$$

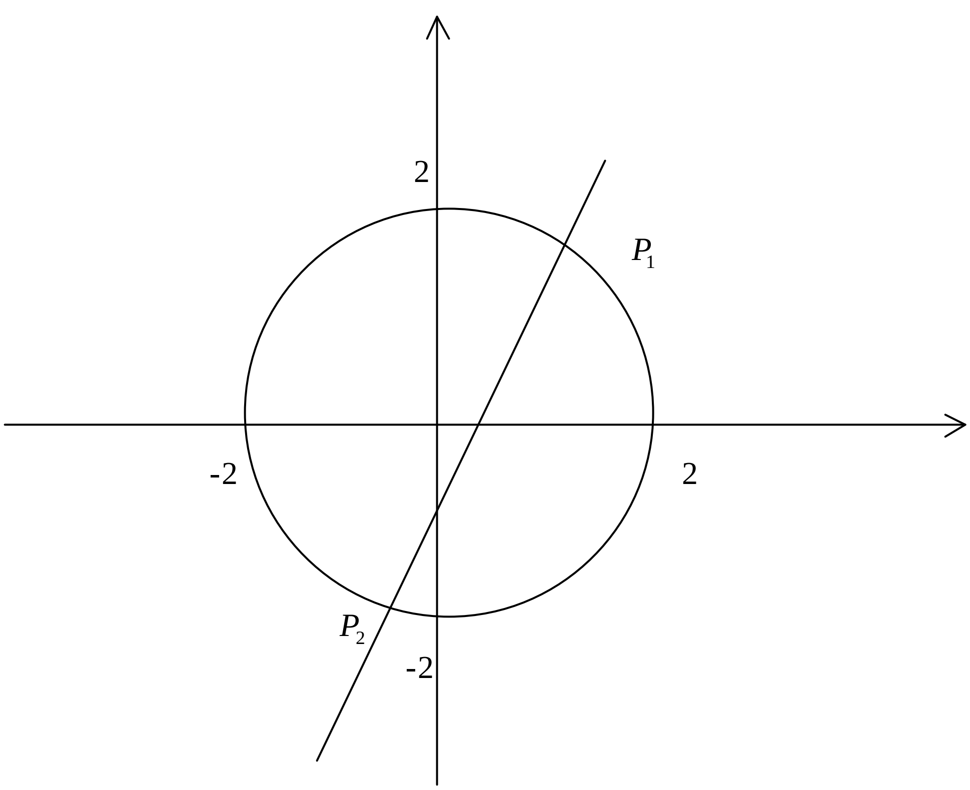
The system has 2 *solutions* $x = 1.27$ *and* $y = 1.54$
$x = -.47$ *and* $y = -1.94$

The geometric interpretation of this is as follows:

Each of the equations $x^2 + y^2 = 4 \quad and \quad -2x + y = -1$ represents a figure in the plane. The first is a circle (of center the origin and radius 2) and the second is a line(of slope 2 and vertical intercept –1)

Each of the solutions of the system is a point in the plane on both figures, that is the points of intersection of the circle and line.

They are $P_1(\,1.27\;,\;1.54\,) \quad and \quad P_2(\,-.47,\;-1.94\,)$



6.

a. 2 x 3
b. 2 x 1
c. 1 x 3
d. 3 x 2
e. 2 x 4
f. 1 x 1

7.

a.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)\,(-1) + 1(1) & 1(2) + (-1)\,1 + 1(3) \\ 2(1) + 0(-1) + 1(1) & 2(2) + 0\,(1) + 1(3) \\ 3(1) + (-1)\,(-1) + 1(1) & 3(2) + (-1)(1) + 1(3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$$

b.

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 + 4 - 5 & 2 + 8 - 10 & 5 + 20 - 25 \\ 2 + 8 - 10 & 4 + 16 - 20 & 10 + 40 - 50 \\ -1 - 4 + 5 & -2 - 8 + 10 & -5 - 20 + 25 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is a case where AxB=0 and A?0 and B?0.

Clearly the algebra of matrices is unlike that of numbers!

8.

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$A^2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \times \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ab + ba \\ ba + ab & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{pmatrix}$$

$$Now\ A^2 + A = \begin{pmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + a + b^2 & 2ab + b \\ 2ab + b & a^2 + a + b^2 \end{pmatrix}$$

$$Now\ A^2 + A = 0 \Rightarrow a^2 + a + b^2 = 0$$

$$2ab + b = 0$$

$$But\quad 2ab + b = (2a + 1)b = 0 \Rightarrow a = -\frac{1}{2}\ \ or\ \ b = 0$$

$$Now\ a = -\frac{1}{2} \Rightarrow \frac{1}{4} - \frac{1}{2} + b^2 = 0$$

$$so \qquad b^2 = \frac{1}{4}$$

$$and \qquad b = \frac{1}{2}\ \ or - \frac{1}{2}$$

Two solutions are

a	b
$-\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$

But $b=0 \quad \Rightarrow \quad a^2+a+0=0$

$$a^2+a=0$$

$$a(a+1)=0$$

so $a=0$ *or* $a=-1$

Two more solutions are

a	b
0	0
-1	0

There are four solutions :

a	b
$-\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$
0	0
-1	0

9.

x+2y+3z=3

2x+5y+7z=6

3x+7y+8z=5

$$A=\begin{pmatrix}1&2&3\\2&5&7\\3&7&8\end{pmatrix},\quad B=\begin{pmatrix}3\\6\\5\end{pmatrix}\quad X=\begin{pmatrix}x\\y\\z\end{pmatrix}\quad so\quad AX=B.$$

$$\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&7&0&1&0\\3&7&8&0&0&1\end{array}\right]\xrightarrow{R_2\to R_2-2R_1,\ R_3\to R_3-3R_1}\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&1&-2&1&0\\0&1&-1&-3&0&1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-2R_2,\ R_3\to R_3-R_2}\left[\begin{array}{ccc|ccc}1&0&1&5&-2&0\\0&1&1&-2&1&0\\0&0&-2&-1&-1&1\end{array}\right]$$

$$\xrightarrow{R_3\to -\frac{1}{2}R_3}\left[\begin{array}{ccc|ccc}1&0&1&5&-2&0\\0&1&1&-2&1&0\\0&0&1&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-R_3,\ R_2\to R_2-R_3}\left[\begin{array}{ccc|ccc}1&0&0&4.5&-2.5&.5\\0&1&0&-2.5&.5&.5\\0&0&1&.5&.5&-.5\end{array}\right]$$

$$So \quad A^{-1} = \begin{bmatrix} 4.5 & -2.5 & .5 \\ -2.5 & .5 & .5 \\ .5 & .5 & -.5 \end{bmatrix}$$

Now ⧖ $= A^{-1}B = \begin{bmatrix} 4.5 & -2.5 & .5 \\ -2.5 & .5 & .5 \\ .5 & .5 & -.5 \end{bmatrix} \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix} = \dfrac{1}{2} \begin{bmatrix} 9 & -5 & 1 \\ -5 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}$

⧖ $= \dfrac{1}{2} \begin{pmatrix} 27 & -30 & +5 \\ -15 & +6 & +5 \\ 3 & +6 & -5 \end{pmatrix} = \dfrac{1}{2} \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

So x =1, y = - 2, and z =2.

10.

x+y+z=1

.2x+.4y+.5z=.45

x+2y+5z=2.80

A= $\begin{pmatrix} 1 & 1 & 1 \\ .2 & .4 & .5 \\ 1 & 2 & 5 \end{pmatrix}$, B= $\begin{pmatrix} 1 \\ .45 \\ 2.80 \end{pmatrix}$

⧖ $= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad AX = B$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ .2 & .4 & .5 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - .2R_1,\ R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & .2 & .3 & -.2 & 1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to 5R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1.5 & -1 & 5 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2\ ,\ R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -.5 & 2 & -5 & 0 \\ 0 & 1 & 1.5 & -1 & 5 & 0 \\ 0 & 0 & 2.5 & 0 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{2.5} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -.5 & 2 & -5 & 0 \\ 0 & 1 & 1.5 & -1 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & .4 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + \frac{1}{2} R_3\ ,\ R_2 \to R_2 - 1.5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -6 & .2 \\ 0 & 1 & 0 & -1 & 8 & -.6 \\ 0 & 0 & 1 & 0 & -2 & .4 \end{array}\right]$$

$$So\ \ A^{-1} = \begin{pmatrix} 2 & -6 & .2 \\ -1 & 8 & -.6 \\ 0 & -2 & .4 \end{pmatrix}$$

Now

⧖ $= A^{-1}B = \begin{pmatrix} 2 & -6 & .2 \\ -1 & 8 & -.6 \\ 0 & -2 & .4 \end{pmatrix} \begin{pmatrix} 1 \\ .45 \\ 2.80 \end{pmatrix}$

$$= \begin{pmatrix} 2 & -2.7 & +5.6 \\ -1 & +3.6 & -1.68 \\ 0 & -.90 & +1.12 \end{pmatrix} = \begin{pmatrix} -.14 \\ .92 \\ .22 \end{pmatrix}$$

So x = -.14, y = .92, and z = .22

The proportions are short A (14%), long B (92%) and long C (22%). Indeed if the value of the investment is $100,000 then $92,000 should be in B, $ 22,000 in C and the trader should be short $ 14,000 of A. This does not take into account margin requirements or other requirements.