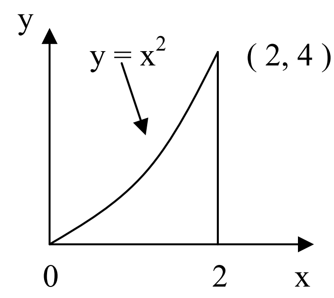
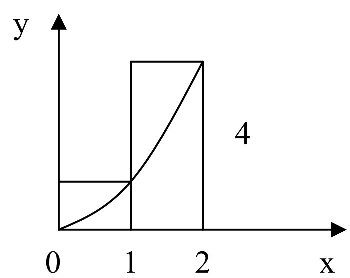
Lesson 9
Integral Calculus
Applications

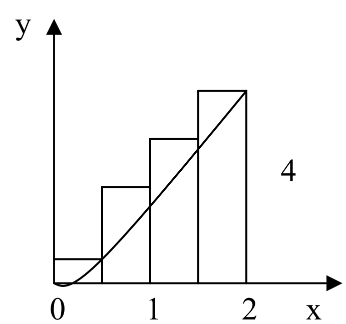
What is the area of a region bounded by curves? Consider the following example:

The area under the graph of $y = x^2$
from x = 0 to x = 2.





Begin by approximating the area with 2 rectangles.
The rectangular approximation has an area that is the sum of the areas of the rectangles: 1 + 4 = 5.
Clearly, this method of approximation overstates the area.



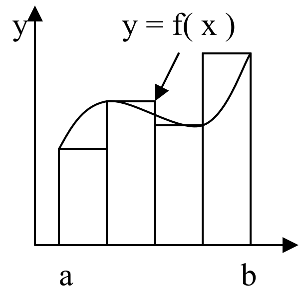
Now, let's approximate the region with 4 rectangles.
As before the rectangular approximation has an area of .125 + .5 + .1.125 + 2 = 3.75

This is a better estimate because there is less excess area.
Continuing on this way we find:

n = number of rectangles	2	4	10	50	100	1000
A = total area of rectangles	5	3.75	3.08	2.75	2.7	2.667

That the area under the curve is 2.667 or 8/3.

More generally consider a region as per figure
To find the area of the region bounded by f, the x-axis, x = a, and x = b proceed as before. Each rectangle has width of (b – a) / n and height f(x_j). For an appropriate x_j, such that $a \le x_j \le b$.



Definition: The Definite Integral

$$A = \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j)\left(\frac{b-a}{n}\right)$$

The Fundamental Theorem of Calculus

If f is a continuous function for $a \le x \le b$ then:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Ex. 1:

$$\int_0^2 x^2 = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = 8/3$$

Ex. 2:

$$\int_0^5 e^x = e^x\Big|_0^5 = e^5 - e^0 = 148.41 - 1 = 147.41$$

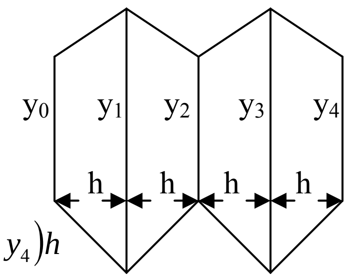
Notice that $\int_a^a f(x)dx = 0$

The Trapezoidal Rule

Trapezoids are much more efficient for estimating the area under a curve. What is the area of a trapezoid?

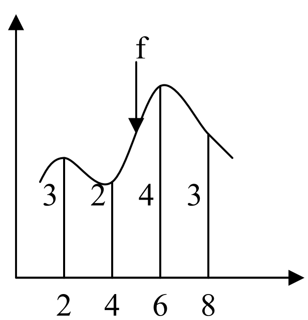
$$A = \frac{1}{2}(a+b)\cdot h$$

What is the area of contiguous trapezoids?



$$\begin{aligned}A &= \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \frac{1}{2}(y_2 + y_3)h + \frac{1}{2}(y_3 + y_4)h \\ &= \frac{1}{2}h\left[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + y_3 + y_4\right] \\ &= \frac{1}{2}h\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right]\end{aligned}$$

What is $\int_2^8 f$?



The rectangular approximation and the trapezoidal approximation are approximately the the same !

Now, n = 3 and h = 2

$$\int_2^8 f \ \underset{=}{a}\ \frac{1}{2}(2)[3 + 2(2) + 2(4) + 3] = 1[18] = 18$$

Rule: $\int_a^b f = \frac{1}{2}\left(\frac{b-a}{n}\right)\left[f(x_0) + \sum_{j=1}^{n-1} 2.\,f(x_j) + f(x_n)\right]$

There are n trapezoids , each with width $\frac{b-a}{n}$

$\int_a^b f$ is most easily found by using the fundamental theorem of calculus. If finding *F* is difficult or a formula doesn't exist (and this can happen) for *F* then the rectangular approximation can be used. But trapezoids are more efficient than the rectangular approximation.

Finding the area under the **Standard Normal Curve**

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

from x = 0 to x = 1.

Because the function is not integratable, we use the trapezoidal rule with n = 8 trapezoids, so that the width h of each trapezoid is .125. The table is:

X	0	.125	.25	.375	.5	.625	.750	.875	1.0
Y	.3989	.3958	.3866	.3718	.3520	.3281	.3011	.2720	.2419

So $\int_0^1 \frac{1}{\sqrt{2\Pi}} e^{-.5x^2} =$

$$\frac{1}{2}(.125)[.39894 + 2(.39584) + 2(.37186) + 2(.38667) + 2(.35207) + 2(.30114) + 2(.27205) + .24197]$$

$$= \frac{1}{2}(.125)[5.4569] = .34103$$

In fact, the entire area under the standard normal curve from -∞ to +∞ is 1. This is because the standard normal is a **probability density function** (pdf). A probability density function is a function that describes the relative likelihood, or probability, for a random number to be drawn from within any range in the observation space.

Probability Density Function

Given a function *f* (*x*), then the function is a density if and only if:

$$\int_a^b f(x)dx = 1$$

Ex. 1:

Given the function $f(x) = kx$, $0 \le x \le 3$. What is the value of *k* if *f* is to be probability density?

$$\int_0^3 kxdx = 1$$

$$\frac{k}{2}x^2 \Big|_0^3 = 1$$

$$\frac{9k}{2} = 1$$

$$k = \frac{2}{9}$$

The Expected Value

The expected value of a function is the arithmetic average or mean, usually denoted by E(x) or μ The mean is a measure of central tendency of the distribution.

$$E(x) = \mu = \int_a^b x \cdot f(x)dx$$

The Variance

The variance of a probability distribution is a measure of dispersion around the mean.

$$V(x) = \sigma^2 = \int_a^b x^2 \cdot f(x)dx - \mu^2$$

Ex. 2:

Given the probability density function $f(x) = \frac{2}{9}x$, for $0 \le x \le 3$, calculate the expected value and variance.

$$E(x) = \int_0^3 x \cdot \frac{2}{9}xdx = \int_0^3 \frac{2}{9}x^2 dx = \frac{2}{27}x^3 \Big|_0^3 = 2$$

$$V(x) = \int_0^3 x^2 \cdot \frac{2}{9}xdx - E(x)^2 = \int_0^3 \frac{2}{9}x^3 dx - 4 = \frac{2}{36}x^4 \Big|_0^3 - 4 = \frac{162}{36} - 4 = .5$$

Project 9

Find the area under the curve:

1. $f(x) = 2e^x \quad 0 \le x \le 2$

2. $f(x) = .25x^2 \quad 1 \le x \le 3$

3. $f(x) = 5 \quad 4 \le x \le 10$

Find the value of *k* that makes each function a density:

4. $f(x) = k \cdot \frac{1}{2} x^2 \quad 0 \le x \le 2$

5. $f(x) = ke^{2x} \quad 0 \le x \le 3$

6. $f(x) = k(2x+2)^2 \quad 0 \le x \le 1$

Find the expected value and variance of each function.

7. $f(x) = k \cdot 3x^{1/2} \quad 0 \le x \le 4$

8. $f(x) = k \cdot e^{1/2x} \quad 0 \le x \le 2$

9. $f(x) = 4k \quad 0 \le x \le 3$