Lesson 7
Integral Calculus
Antiderivatives

Definition: A function F is an antiderivative of f if and only if:

$$F'(x) = f(x)$$

The family of antiderivatives of a function f is denoted by:

$$F(x) = \int f(x)dx$$

Or,

$$f(x) = \int f'(x)dx$$

The notation is a little difficult to follow. Consider this example:

$$f'(x) = \frac{dy}{dx} = x^2 + 2$$

We could otherwise write this using the slope-point equation of a line. The change in y is equal to the rate of change in y given a change in x, times the change in x:

$$dy = (x^2 + 2)dx$$

Now, if we integrate both sides, the left side with respect to y and the right side with respect to x we get:

$$\int dy = \int (x^2 + 2)dx$$

On the left hand side, as you will see, via the rules of integration, we end up with:

$$\int 1dy = y$$

So that,

$$y = \int (x^2 + 2)dx$$

In general then,

$$y = f(x) = \int f'(x)dx$$

Consider the following table:

f(x)	$x^2 - 4.3$	$x^2 - 1$	x^2	$x^2 - 3047$
f '(x)	$2x$	$2x$	$2x$	$2x$

Rule: If $F' = f$ and $G' = f$, then $G = F + c$ or $G - F = c$ where c is some constant. All antiderivatives of a function differ by an additive constant, i.e. if you know one antiderivative you know them all.

Consider the model of position, time, and velocity. If two particles have the same velocity at every instant, then the distance between them is unchanging, i.e constant.

Ex. 1: If $f(x) = 2x$, then $F(x) = \int 2x\,dx = x^2 + c$

Finding antiderivatives is the reverse of differentiating.

Rule 1: If k is a constant, then $\int k\,dx = kx + c$

Rule 2: $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$

Rule 3: $\int x^{-1} dx = \ln(x) + c$

Rule 4: $\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + c$

Rule 5: If k is a constant, then $\int kf(x)dx = k\int f(x)dx$

Rule 6: $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Rule 7: $\int k^{f(x)} dx = \frac{1}{\ln(k)} k^{f(x)} + c$

Rule 8: $\int \frac{1}{f(x)} dx = \ln(f(x)) + c$

Integration by Parts: $\int f(x) \cdot g'(x) = f(x) \cdot g(x) - \int f'(x) \cdot g(x)$

Be careful:

$$\int (f(x) \cdot g(x))dx \neq \int f(x)dx \cdot \int g(x)dx$$

Ex. 2: $\int -7dx = -7x + c$

Ex. 3: $\int 4.3dx =$

Ex. 4: $\int x^2 dx = \frac{1}{3}x^3 + c$

Ex. 5: $\int x^3 dx =$

Ex. 6: $\int \sqrt{x} dx =$

Ex. 7: $\int \sqrt[4]{x} dx =$

Ex. 8: $\int \frac{1}{x^5} dx =$

Ex. 9: $\int e^{3x} dx = 3 \cdot \int e^{3x} \cdot \frac{1}{3} dx = 3e^{3x} + c$

Ex. 10: $\int \frac{1}{e^{2x}} dx =$

Ex. 11: $\int 2x^5 dx =$

Ex. 12: $\int (2x^3 + 5x^2 - 7x + 8)dx =$

Ex. 13: $\int \left(\frac{4}{\sqrt[3]{x^2}} - 8 \cdot \sqrt[4]{x^3} + 1 \right) dx =$

Ex. 14: $\int e^{x^2} dx =$

Ex. 15: $\int xe^{x^2} dx =$

Ex. 16: $\int e^{4x+1} dx =$

Ex. 17: $\int (5x+7)^3 dx = \frac{1}{5} \cdot \int (5x+7)^3 \cdot 5dx =$

Ex. 18: $\int (2x^3 + 3x)^4 x^2 dx =$

Ex. 19: $\int \frac{x}{x^2+1} dx = \frac{1}{2} \cdot \int \frac{2x}{x^2+1} dx =$

Ex. 20: $\int \frac{2x^4}{4x^5 - 7} dx =$

Ex. 21: $\int \frac{1}{x} dx =$

Ex. 22: $\int \ln(x) dx =$

Ex. 23: $\int \frac{1}{\sqrt{4-3x}} dx =$

Ex. 24: $\int \sqrt[3]{1-2x} dx =$

Ex. 25: $\int 2^{3x} dx =$

Example: A particle is moving so that $a_t = 60t^3 - 12t + 4$ and $V_0 = 10$ and $s_0 = 8$. What is the position at $t = 2$.

$$V_t = \frac{60t^4}{4} - \frac{12t^2}{2} + 4t + c = 15t^4 - 6t^2 + 4t + c$$

Now,

$$10 = V_0 = 15(0)^4 - 6(0)^2 + 4(0) + c = c \Rightarrow c = 10$$

So,

$$V_t = 15t^4 - 6t^2 + 4t + 10$$

Now,

$$s_t = \frac{15t^5}{5} - \frac{6t^3}{3} + \frac{4t^2}{2} + 10t + c$$

$$s_t = 3t^5 - 2t^3 + 2t^2 + 10t + c$$

So,

$$8 = s_0 = 3(0)^5 - 2(0)^3 + 2(0)^2 + 10(0) + c = c$$

And,

$$s_t = 3t^5 - 2t^3 + 2t^2 + 10t + 8$$

Whence,

$$s_2 = 3(2)^5 - 2(2)^3 + 2(2)^2 + 10(2) + 8 = 116$$

Example: If $f'(x) = x^2 - 2$ and $(1,-1)$ is on the graph of *f*, then find *f*.

Definition: If all else fails, use integration by parts:

$$(f \cdot g)' = f' \cdot g + f \cdot g$$

If must be the case then that:

$$\int (f \cdot g)' dx = \int (f' \cdot g + f \cdot g') dx$$

So,

$$f \cdot g = \int f' \cdot g dx + \int f \cdot g' dx$$

And,

$$\int f \cdot g' dx = f \cdot g - \int f' \cdot g dx$$

Example: $\int x e^x dx =$

Example: $\int x \ln(x) dx$

Project 7

For each of the following functions , f, find the family of antiderivatives , $\int f$

1. $y = \sqrt[3]{x^4}$
2. $y = 5x^9 - \dfrac{2}{x^7}$
3. $y = \dfrac{x^2 + 2x + 1}{x}$
4. $y = x\left(x^3 - 2x\right)$
5. $y = 1/7x$
6. $y = -4e^{6x}$
7. $y = (3x - 1)^4$
8. $y = x\left(3x^2 + 2\right)^{2/3}$
9. $y = (\ln x).\dfrac{1}{x}$
10. $y = \dfrac{3x^2}{1 + 2x^3}$
11. $y = x^2 e^{x^3 + 1}$
12. $y = (1.7)^x$
13. $y = \dfrac{e^x}{1 + e^x}$
14. $y = x^{.3} + 4$

15. If the velocity of a particle is given by $V_t = 4\sqrt{t}$ and $s_0 = 10$ then find s_8 , the position at t = 8.

16. If $f''(x) = -32$ and $f'(1) = 12$ and $f(2) = 36$ then find f(3).

17. Use integration by parts to find $\int \sqrt{x}\ln(x)$

18. Use integration by parts and the fact that ln(x)=1·ln(x) to find $\int \ln(x)$.