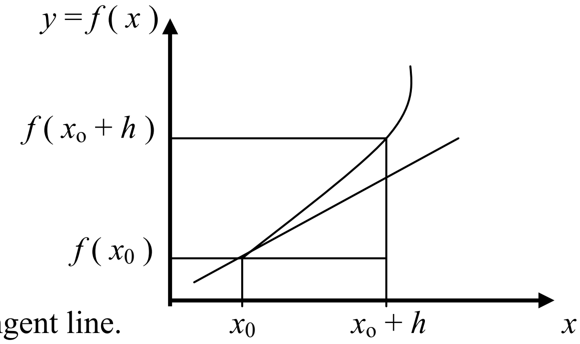
# Lesson 6
## Applications of Differential Calculus

The line tangent to the graph of a function $f$ at $( x_0, f( x_0 ) )$ is the line passing through the point whose slope is $f'( x_0 )$.




The slope-point equation of the line is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The tangent line is the geometry of the derivative. It is the line that kisses f at $( x_0, f( x_0 ) )$. Each point on $f( x )$ has a tangent line.

Find the equation of the tangent line to $y = f( x ) = x^2$ at ( 1, 1 ).

$$y - f( 1 ) = f'( 1 )( x - 1 )$$
$$y - 1 = 2 ( x - 1 )$$
$$y - 1 = 2x - 2$$
$$y = 2x - 1$$

**Ex. 1:** Find the equation of the tangent line to $y = f( x ) = e^x$ at ( 0, 1 ).

Clearly, $f( x ) = f( x_0 ) + f'( x_0 )( x - x_0 ) + \varepsilon$

$f( x ) - f( x_0 ) = f'( x_0 )( x - x_0 ) + \varepsilon$

That is, more generally, we can estimate the change in $y$ as being equal to the derivative times the change in $x$.

**Rule:** $\Delta y = f'( x_0 ) \cdot \Delta x + \varepsilon$, when $\Delta x$ is small

**Ex. 2:** If $s = f( t )$ and $f'( t_2 ) = v_2 = 50$ miles / hr, then $\Delta t = .1$ implies that $\Delta s = 50 \cdot .1 = 5$ miles. So, if you are driving at 50 miles per hour, in the next 1 / 10 of an hour you will drive 5 miles, *as long as your speed does not change.*

**Taylor’s Theorem:**

Taylor's theorem gives an approximation of a differentiable function near a given point by polynomials, whose coefficients depend only on the derivatives of the function at that point. Taylor’s theorem lets one approximate a function *f*, around $x_0$, by a quadratic function. The approximation improves as $\Delta x \to 0$. Polynomial functions are mathematically pleasant to work with , hence the motive for approximation.

$$f(x_1) \approx f(x_0) + \frac{1}{1!} f'(x_0)(x_1 - x_0) + \frac{1}{2!} f''(x_0)(x_1 - x_0)^2 + \ldots + \frac{1}{n!} f^n(x_0)(x_1 - x_0)^n$$

In finance, we usually only go to the second term so that:

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2} f''(x_0)(x_1 - x_0)^2$$

**Ex. 3:** Given that $y = f(x) = x^2$ and that $x_0 = 2$, estimate the new value of $y$ when $\Delta x = 1$.

| | |
|---|---|
| $f(x) = x^2$ | $f(2) = 4$ |
| $f'(x) = 2x$ | $f'(2) = 4$ |
| $f''(x) = 2$ | $f''(2) = 2$ |

$$f(x_1) \approx f(2) + f'(2)(1) + \frac{1}{2} f''(2)(1)^2$$

$$f(x_1) \approx 4 + 4 + 1 = 9$$

Note that $f(3) = 9$.

**Ex. 4:** Given the following data:

| | | | | |
|---|---|---|---|---|
| **S** | **Stock price** | 45 | 50 | 55 |
| **C** | **Call price** | 5 | 7 | 8.5 |
| **dC/dS** | **Call delta** | .4 | .6 | .7 |
| $\mathbf{d^2C / dS^2}$ | **Call gamma** | .2 | .1 | .07 |

Using Taylor’s Theorem to two terms, if the price of stock is 50, what is the new price of the call if the stock goes up 1 point?

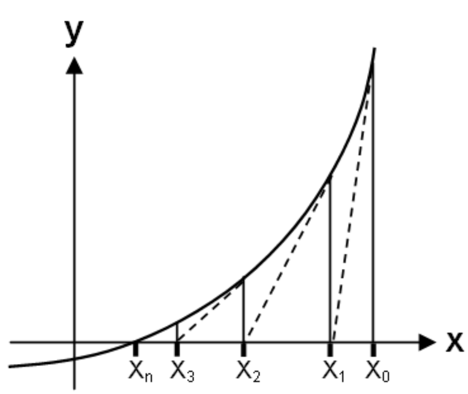
$$f(x_1) \approx 7 + .6 \cdot 1 + \frac{1}{2} \cdot .1 \cdot 1^2 = 7.65$$

**Ex. 5:** Given that $s = f(t)$ and you are driving at a velocity of $f'(t_2) = v_2 = 50$ mi / hr, and your speed is increasing at a rate of acceleration $f'(t_2) = a_2 = 10$ mi / hr$^2$. Then $\Delta t = .1$ implies $\Delta s = 50 \cdot .1 + .5 \cdot 10 \cdot .1^2 = 5.05$ miles

**Ex. 6:** If $S_o = 45$ and the stock decreases by ½ point, then what is the *change in the value* of the call option?

**Newton's Method**

Newton's method uses derivatives and lines tangent to the graph of the function to solve complex problems. The value of the derivative at any $x$ is the slope of the line tangent to the graph of the function at $( x, f( x ) )$. Consider the following graph of a function y = $g( x )$. To solve $g( x ) = 0$, where $x_n$ is the solution, we start with an estimate $x_0$ close to $x_n$. $x_0$ is called the initial guess. Next consider the tangent line to g at $x_0$.




The equation of the tangent line is:

$$y - g(x_0) = g'(x_0)(x - x_0)$$

The tangent line intersects the $x$-axis at $x_1$ where $x_1$ is clearly closer to $x_n$ than $x_0$. It follows then that:

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

Newton's method is this recursive procedure, summarized thusly to find the root(s) of $g( x )$:

1. Make an initial guess, $x_0$
2. Proceed through the recursive procedure.

$$x_{j+1} = x_j - \frac{g(x_j)}{g'(x_j)}$$

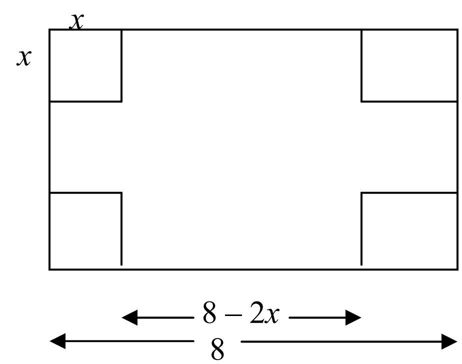
3. At some guess, j, the solution will be close enough to 0 to accept $x_j$ as an approximation to the root.

Different initial guess values may lead to different roots. A unique guess value is necessary for each root.

**Ex. 1:** Betty and Bob Box Company make open topped boxes from a sheet of cardboard that is 6 feet by 8 feet by cutting an identical square from each corner. The volume $V$ depends on the length of the cut $x$ as per

$V = f(x) = x(6 - 2x)(8 - 2x) \quad 0 < x < 3$

$V = 48x - 28x^2 + 4x^3$




If the volume of the box must be 20 cu.ft, then what should $x$ be? Clearly we must solve the equation:

$4x^3 - 28x^2 + 48x - 20 = 0$

Using Newton's Method:

$g(x) = 4x^3 - 28x^2 + 48x - 20$

$g'(x) = 12x^2 - 56x + 48$

| j | $x_j$ | $g(x_j)$ | $g'(x)$ | $g(x_j)/g'(x_j)$ | $x_j - g(x_j)/g'(x_j)$ |
|---|---|---|---|---|---|
| **0** | 1 | 4 | 4 | 1 | 0 |
| **1** | 0 | -20 | 48 | -.41667 | +.41667 |
| **2** | .41667 | -4.572 | 26.750 | -.17092 | +.58758 |
| **3** | .58758 | -.6517 | 19.239 | -.03387 | +.62145 |
| **4** | .62145 | -.0240 | 17.833 | -.00134 | +.62279 |
| **5** | .62279 | -.0001 | | | |

We accept $x \approx .622$ as our approximate solution. Indeed a TI-82 calculator gives .622797146 as the approximate solution.

**Ex. 2:** Find the point of intersection in the first quadrant of $y = e^x$ and $y = 2 - x$.

$$g(x) = e^x + x - 2$$
$$g'(x) = e^x + 1$$

Use $x_0 = 1$ as the initial guess.

| j | $x_j$ | $g(x_j)$ | $g'(x_j)$ | $g(x_j)/g'(x_j)$ | $x_j - g(x_j)/g'(x_j)$ |
|---|---|---|---|---|---|
| **0** | 1 | 1.71828 | 3.71828 | 0.46212 | 0.53788 |
| **1** | 0.53788 | 0.25025 | 2.71237 | 0.09226 | 0.44562 |
| **2** | 0.44562 | 0.00708 | 2.56146 | 0.00276 | 0.44286 |
| **3** | 0.44286 | 0.00001 | 2.55715 | 0.00004 | 0.44286 |

We accept 0.44286 as the solution.

**Ex. 3:** Consider a 1½ year bond with equation

$$100 = \frac{5}{1+R} + \frac{5}{(1+R)^2} + \frac{105}{(1+R)^3}; 0 < R$$

Note the price is 100 and semi-annual coupons are 5. What is the semiannual yield to maturity $R$?

Let $x = \frac{1}{1+R}$ whence $100 = 5x + 5x^2 + 105x^3$

So $g(x) = 105x^3 + 5x^2 + 5x - 100 = 0$ must be solved.

Note $g'(x) = 315x^2 + 10x + 5$. Using Newton's Method with $x_0 = .90$ we get:

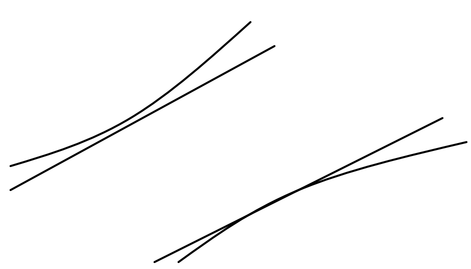
| j | $x_j$ | $g(x_j)$ | $g'(x_j)$ | $g(x_j)/g'(x_j)$ | $x_j - g(x_j)/g'(x_j)$ |
|---|---|---|---|---|---|
| **0** | .90 | -14.91 | 269.75 | -.0554 | .9554 |
| **1** | .9554 | .9092 | 302.08 | +.0030 | .9524 |
| **2** | .9524 | .0057 | 300.25 | .00002 | .9524 |

We accept $\tilde{x} = .9524$. Now $\tilde{x} = \frac{1}{1+\tilde{R}}$ so $1 + \tilde{R} = \frac{1}{\tilde{x}}$ whence

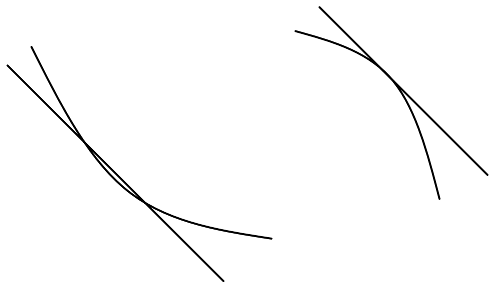
$\tilde{R} = \frac{1}{\tilde{x}} - 1 = \frac{1}{.9524} - 1 = .05$ The bond yields 10% per annum compounded semiannually.

**Maxima and Minima**

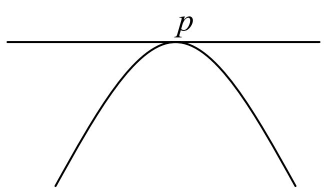
Here are some observations about the relationship of the graph of *f* and the value of *f* ‘
The tangent line provides the key insight!



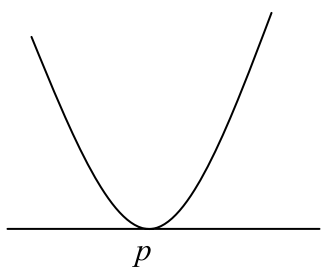
The tangent line has positive slope. i.e. $f' > 0$. Therefore, the function *f* is increasing.



The tangent line has negative slope. i.e. $f' > 0$. Therefore, the function $f$ is decreasing.




The tangent line is horizontal at point $p$, $f'(p) = 0$. The slope to left is + and the slope to the right is –. Therefore, $p$ is a local maximum.




The tangent line is horizontal at point $p$, $f'(p) = 0$. The slope to left is – and the slope to the right is +. Therefore, $p$ is a local minimum.

**Definition:** $x_0$ is called a **critical point** of $f$ if and only if $f'(x) = 0$. Maxima and minima occur at critical points, end points, or points where cusps or corners occur ( $f'$ not defined at these points)

**Ex. 1 Revisited:**

Consider again Betty and Bob's Box Company. Let's find the cut $x_{max}$ that results in the maximum volume $V_{max}$. Recall that:

$$V = 48x - 28x^2 + 4x^3$$

And,

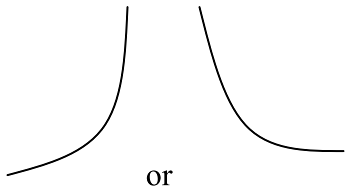
$$V' = 48 - 56x + 12x^2$$

The critical points are the solutions to $V' = 0$. By the quadratic formula the solutions are 3.535 and 1.131. But $0 < x < 3$, so $x = 1.1.31$ is the only critical point. Clearly,

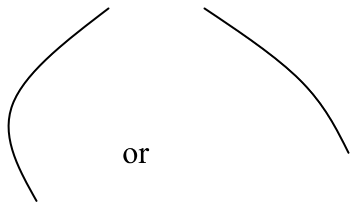
| **$x < 1.131$** | **$x = 1.131$** | **$x > 1.131$** |
|---|---|---|

| $f'(0) = 48$ | $f'(1.131) = 0$ | $f'(2) = -16$ |
|---|---|---|

Just plug in various values of $x$ from the appropriate intervals to confirm the signs of $f'$. Clearly x = 1.131 is an absolute maximum point. Indeed $v_{max} = f(1.131) = 24.258$.




The graph of $f$ is smiling, therefore the second derivative $f'' > 0$.




The graph of $f$ is frowning, therefore the second derivative $f'' < 0$.

**Definition:** A point on the graph of $f$ where $f$ changes from one concavity to another is called an **inflection point**. They occur at the solutions of $f'' = 0$.

**Ex. 5:** Let's analyze the graph of $y = f(x) = x^3 - 9x^2 + 24x - 7$

First we make a partial table of values.

| **x** | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|
| ***y*** | -99 | -41 | -7 | 9 | 13 | 11 | 9 | 13 | 29 |

Next we make an $f'$ analysis.

$f'(x) = 3x^2 - 18x + 24$

$f'(x) = 3(x^2 - 6x + 8)$

$f'(x) = 0$ if and only if $x = 2$ or $x = 4$

These are critical points.

| **x** | $x < 2$ | **2** | $2 < x < 4$ | **4** | $4 < x$ |
|---|---|---|---|---|---|
| ***f'*** | Positive | 0 | Negative | 0 | Positive |
| ***f*** | Increase | Max | Decrease | Min | increase |

So x = 2 is a local maxima and x = 4 is a local minima

Next we make an $f''$ analysis.

$f''(x) = 6x - 18$

$f''(x) = 6(x-3)$
$f''(x) = 0$ if and only if x = 3

| ***x*** | **x < 3** | **3** | **3 < x** |
|---|---|---|---|
| ***f '*** | Negative | 0 | Positive |
| ***f*** | Frowning | Inflex | Smiling |

So x = 3 is an inflection point, frowning to the right, smiling to the left..

**Partial Derivatives**

Consider a right circular cylinder, where the volume, *V*, given by $V = \Pi R^2 L = f(R, L)$, is a function of 2 independent variables. How does *V* change when *R* and *L* are changing? The answer is

$$\Delta V \approx \frac{\partial V}{\partial R} \times \Delta R + \frac{\partial V}{\partial L} \times \Delta L$$

where $\frac{\partial V}{\partial R} = f_R$ and $\frac{\partial V}{\partial L} = f_L$ are the **partial derivatives.**

**Rule:** If $z = f(x, y)$, then to find $\frac{\partial z}{\partial x}$ treat *y* as a constant and find the derivative of *z* with respect to *x*. To find $\frac{\partial z}{\partial y}$ treat *x* as a constant and find the derivative with respect to *y*.

**Ex. 1:** If $z = f(x, y) = e^{x^2+y^2}$ then

$$\frac{\partial z}{\partial x} = e^{x^2+y^2}(2x) \text{ and}$$

$$\frac{\partial z}{\partial y} = e^{x^2+y^2}(2y)$$

**Ex. 2:** $z = f(x, y) = 5x^3y - 10x^2y^2 - 5$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

**Definition**

If $z = f(x, y)$, then $\frac{\partial^2 z}{\partial x^2} = f_{x,x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ is called the 2nd partial derivative,

$\frac{\partial^2 z}{\partial y^2} = f_{y,y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$ is the 2nd partial derivative, and

$\frac{\partial^2 z}{\partial x \partial y} = f_{x,y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ is the mixed partial derivative.

As a matter of fact, for most functions $f_{x,y} = f_{y,x}$ that is, the order of differentiation does not affect the result!

**Ex. 3:** $z = x\ln(y) + y^2 \ln(x)$

$$\frac{\partial z}{\partial x} = \ln(y) + y^2\left(1/x\right)$$

$$\frac{\partial^2 z}{\partial x^2} = y^2\left(-\frac{1}{x^2}\right) = -\frac{y^2}{x^2}$$

$$\frac{\partial z}{\partial y} = x\left(1/y\right) + 2y\ln(x)$$

$$\frac{\partial^2 z}{\partial y^2} = x\left(\frac{-1}{y^2}\right) + 2\ln(x)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\ln(y) + y^2/x\right) = \frac{1}{y} + \frac{2y}{x}$$

**Ex. 4:** $z = 5x^3y - 10x^2y^2 - 5$. Find the second partial derivatives and the mixed partial derivative.

**Project 6**

1. Find the equation of the tangent line to f at the stated x.

a. $f(x) = \ln(x); x = 1$

b. $f(x) = \frac{1}{x}; x = 1$

c. $f(x) = \sqrt{x}$;x=1

d. $f(x) = x^3$;x=0

2. Approximate the function f about the given $x_0$ by Taylor's Theorem with a third degree polynomial

a. $f(x) = e^x, x_0 = 0$

b. $f(x) = \ln(x); x_0 = 1$

c. $f(x) = xe^x; x_0 = 0$

d. $f(x) = \sqrt{x}; x_0 = 1$

3. A stock option position with value ,W=f(x), where x is the stock price has the associated information.

| X | 45 | 50 | 55 |
|---|---|---|---|
| W | 4 | 10 | 2 |
| $dw/dx$ (delta) | 3 | -2 | -5 |
| $d^2w/dx^2$ (gamma) | -2 | -1 | -3 |

Using Taylor's Theorem:

a. If x increases from 50 to 50½ approximate the new value of W. How much is the profit or loss?
b. If x decreases from 50 to 49 approximate the new value of W. How much is the profit or loss?

4. Use Newton's Method to solve the equation:

$$2^x + 3^x = 1{,}000{,}000$$

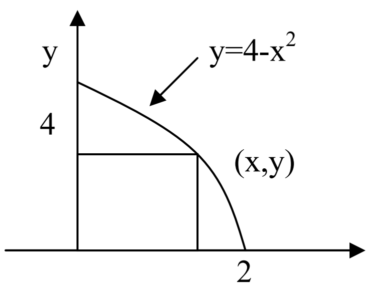
5. Consider a 2 year bond with equation:

$$90 = \frac{5}{1+R} + \frac{5}{(1+R)^2} + \frac{5}{(1+R)^3} + \frac{105}{(1+R)^4}; 0 < R$$

Using the substitution $x = \frac{1}{1+R}$ as per lesson and Newton's Method solve for R as per table.

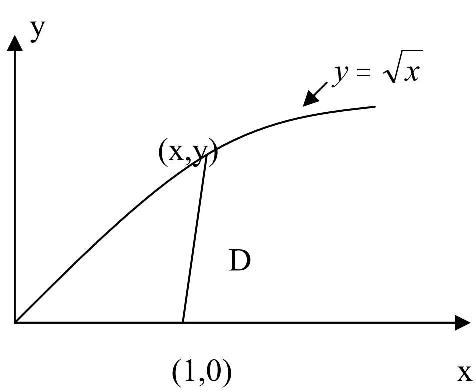
6. For the equation $xe^x = 1{,}000$ use Newton's Method with an initial guess solution of $x_0 = 5$ to find the next approximate solution $x_1$.

7. A rectangle is inscribed as per the diagram:




Find the dimensions of the rectangle of maximum area. What is this area? Justify all of your work by calculus.

8.




Find the point on the curve $y = \sqrt{x}$ that is closest to the point (1,0). What is this minimum distance ,$D_{min}$ ? Justify all your work by calculus. (Do you recall the distance formula?)

9. Analyze as per the lesson the graph of $y = x^3 - 6x^2 + 9x + 6$

10. Analyze as per the lesson the graph of $y = x^4 - 6x^2$.