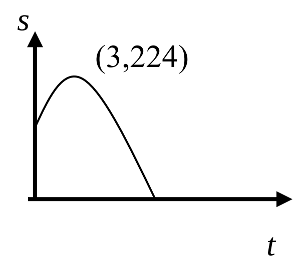
Lesson 5
Differential Calculus

Betty and Bob throw a ball upwards from the top of an 80 foot tower. The position *s* in feet above the ground is a function of the time *t* in seconds:

$$s = f(t) = -16t^2 + 96t + 80$$



t	0	1	2	3	4	5
s	80	160	208	224	208	160

Indeed *f* is a quadratic function and (3, 224) is the maximum point. Clearly, as *t* changes, *s* changes. What is the nature of this change? We start by looking at the **interval velocity** *v* from *t* =1 to *t* = 3:

$$v_{1,3} = \frac{\Delta s}{\Delta t} = \frac{f(3) - f(1)}{3-1}$$

$$v_{1,3} = \frac{224-160}{3-1} = 32 \ \ feet/\sec.$$

What happens if we let the change in time become smaller, say from *t* = 1 to *t* = 2?

$$v_{1,2} = \frac{208-160}{2-1} = 48$$

Now, say from *t* = 1 to *t* = 1.9?

$$v_{1,1.9} = \frac{28}{.5} = 56$$

Now, say from *t* = 1 to *t* = 1.1?

$$v_{1,1.1} = \frac{6.24}{.1} = 62.4$$

Now, say from *t* = 1 to *t* = 1.01?

$$v_{1,1.01} = \frac{.638}{.01} = 63.8$$

What is v_1, the exact or instantaneous velocity at *t* = 1? Clearly, as $\Delta t \to 0$, $v_{1,1+\Delta t} \to 64$. So, $v_1 = 64$. In general, what is the interval velocity from *t* to *t* + *h*?

$$v_{t,t+h} = \frac{\Delta s}{\Delta t} = \frac{f(t+h) - f(t)}{h}$$

$$v_{t,t+h} = \frac{\left(-16(t+h)^2 + 96(t+h) + 80\right) - \left(-16t^2 + 96t + 80\right)}{h}$$

$$v_{t,t+h} = \frac{-32th - 16h^2 + 96h}{h}$$

$$v_{t,t+h} = -32t - 16h + 96$$

What is v_t, the exact or instantaneous velocity at t? As $h \to 0$,

$$v_t = -32t + 96$$

So it is that:

$$v_1 = -32 \cdot (1) + 96 = 64$$

Definition:

If $y = f(x)$, then the instantaneous velocity of y given a change in x is the

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This is the exact rate of change of y given a change in x. The rate of change in y given a change in x is called the **derivative** of f with respect to x. We use many notations to refer to the derivative:

$$\dot{y} = y' = \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note that f' is a function, and $f'(x)$ is a number.

Example:

Let $f(x) = x^2$. Find $f'(x)$ and $f'(3)$.

Examples:

For a bond, the price p is a function of interest rate r. Thus $p = f(r)$. This gives rise to:

$$\frac{dp}{dr} = dollar\ duration$$

For a call option, the price p is a function of the price of the underlying stock s. Thus, $c = f(s)$. This gives rise to:

$$\frac{dc}{ds} = call\ delta$$

Rules of Differentiation

1. **If $y = c$, then $y' = 0$.**

Ex. 1: $y = -6.4.$ $\quad y' = 0$
$y = 3.141$ $\quad y' = 0$
$y = 5.3$ $\quad y' = 0$

2. **If $y = x^r$, then $y' = rx^{r-1}$**

Ex. 2: $y = x$ $\quad y' = 1$
$y = x^2$ $\quad y' = 2x$
$y = x^3$ $\quad y' =$
$y = x^{-1}$ $\quad y' = -1x^{-2}$
$y = x^{-2}$ $\quad y' =$
$y = x^{1/2}$ $\quad y' = (1/2)x^{-1/2}$
$y = x^{-3/2}$ $\quad y' =$

3. **If $y = k \cdot f(x)$, then $y' = k \cdot f'(x)$**

Ex. 3: $y = 5x^2$ $\quad y' = 5 \cdot 2x = 10x$
$y = 14 \cdot x^{1/2}$ $\quad y' =$
$y = 6 \cdot x^{-4}$ $\quad y' =$

4. **If $y = f(x) \pm g(x)$, then $y' = f'(x) \pm g'(x)$**

Ex. 4: $y = mx + d$ $\quad y' = m$
$y = 8x^2 - 7x$ $\quad y' =$
$y = 15x^4 - 9x^3 - 3x^2 + 6$ $\quad y' =$
$y = 3x^2 + 4x^{-3}$ $\quad y' =$

5. **If $y = e^{g(x)}$, then $y' = e^{g(x)} \cdot g'(x)$**

Ex. 5: $y = e^{mx+d}$ $\quad y' = e^{mx+d} \cdot (mx + d)' = e^{mx+d} \cdot m$
$y = e^{x^2+1}$ $\quad y' = e^{x^2+1} \cdot 2x$
$y = 4e^{x^3}$ $\quad y' =$

6. **If $y = \ln(g(x))$, then $y' = g'(x) / g(x)$**

Ex. 6: $y = \ln(2x^3 + 5)$ $\quad y' = 6x^2 / 2x^3 + 5$
$y = \ln(1 - 3x + 4x^2)$ $\quad y' =$

7. If $y = f(x) \cdot g(x)$, then $y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ The Product Rule

Ex. 7: $y = xe^x$ $\quad y' = 1 \cdot e^x + x \cdot e^x$
$\quad = (1 + x)\, e^x$

$f(x) =$	x	$g(x) =$	e^x
$f'(x) =$	1	$g'(x) =$	e^x

$y = x \cdot \ln(x)$ $\quad y' =$

8. If $y = f(x) / g(x)$, then $y' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$ **The Quotient Rule**

Ex 8: $y = (2x - 4) / (3x + 5)$

$f(x) =$	$2x - 4$	$g(x) =$	$3x + 5$
$f'(x) =$	2	$g'(x) =$	3

$$y' = \frac{2 \cdot (3x+5) - 3 \cdot (2x-4)}{(3x+5)^2} = \frac{22}{(3x+5)^2}$$

$y = 2x / (x^2 + 1)$ $\quad y' =$

9. If $y = (g(x))^r$, then $y' = r \cdot (g(x))^{r-1} \cdot g'(x)$ The Generalized Power Rule

Ex. 9: $y = (5x + 1)^3$ $\quad y' = 3 \cdot (5x + 1)^2 \cdot (5x + 1)'$
$\quad = 3 \cdot (5x + 1)^2 \cdot 5$
$\quad = 15 \cdot (5x + 1)^2$

$y = \sqrt{2x+1}$ $\quad y' =$

10. If $y = k^{f(x)}$, then $y' = \ln(k) \cdot k^x \cdot f'(x)$

Ex. 10: $y = 2^x$ $\quad y' = \ln(2) \cdot 2^x = .693\,(2^x)$
$y = 3^{4x+4}$ $\quad y' =$

Examples:

1. If $y = e^{1/x}$ then

2. If $y = \frac{e^x}{1+e^x}$ then

3. If $y = (x-1)^3 x^2$ then

4. If $y = \ln\left(\sqrt[n]{x^m}\right)$ then

5. If $y = 4x^3 - 5x^2 + 7x$

then $\frac{dy}{dx} = 12x^2 - 5x + 7$

and $\frac{dy}{dx}(2) = 12(2)^2 - 5(2) + 7$

= 48-10+7

= 45

6. If $y = e^{3x}$ then find $y'(0)$

The Second Derivative

Since f' is a function, it makes sense to find the derivative of f'. This is called the second derivative of f and denoted by:

$$\ddot{y} = y'' = \frac{d^2y}{dx^2} = f''(x)$$

If $s = f(t)$, then $v_t = \frac{ds}{dt}$ is the velocity, and $a_t = \frac{d^2s}{dt^2}$ is the acceleration.

If $p = f(r)$ for a bond, then $\frac{d^2p}{dr^2}$ is the dollar convexity.

If $c = f(s)$ for a call option, then $\frac{d^2c}{ds^2}$ is the call gamma.

If $y = x^5$ then $y' = 5x^4$ and $y'' = 20x^3$ and $y''' = 60x^2$ and $y^{(4)} = 120x$ and $y^{(5)} = 120$ and $y^{(6)} = 0$

Summary of Rules

1. **If $y = c$, then $y' = 0$.**

2. **If $y = x^r$, then $y' = rx^{r-1}$**

3. **If $y = k \cdot f(x)$, then $y' = k \cdot f'(x)$**

4. **If $y = f(x) \pm g(x)$, then $y' = f'(x) \pm g'(x)$**

5. **If $y = e^{g(x)}$, then $y' = e^{g(x)} \cdot g'(x)$**

6. **If $y = \ln(g(x))$, then $y' = g'(x) / g(x)$**

7. **If $y = f(x) \cdot g(x)$, then $y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$** **The Product Rule**

8. **If $y = f(x) / g(x)$, then** $y' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$ **The Quotient Rule**

9. **If $y = (g(x))^r$, then $y' = r \cdot (g(x))^{r-1} \cdot g'(x)$** **The Generalized Power Rule**

10. **If $y = k^{f(x)}$, then $y' = \ln(k) \cdot k^x \cdot f'(x)$**

Project 6

Find the derivative of each of the following functions

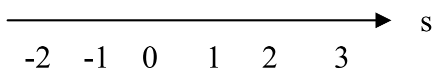
1. $y = 5 - 2x^2 + x^4$
2. $y = 4x^3 + 2 + \frac{1}{x}$
3. $y = 2\sqrt{x} - \frac{3}{\sqrt[3]{x}}$
4. $y = \frac{1}{\sqrt[3]{x^2}}$
5. $y = (x+1)(x^3+1)$
6. $y = \frac{3-x}{x^2-3}$
7. $y = xe^{-x^2}$
8. $y = \ln(1+e^x)$
9. $y = \sqrt{\ln(x)}$
10. $y = \frac{e^{x^2}}{e^x}$
11. $y = \ln(3)\ln(x)$
12. $y = 3x^5 + 7x^3 - 4x^2 + 12$
13. $y = x^{\sqrt{2}}\ln(x)$
14. $y = \frac{\ln(x)}{x}$
15. $y = (1+\sqrt{x})(3-2x)$
16. $y = \frac{8}{x+1}$
17. $y = \frac{5x^2 - 7x + 4}{x}$
18. $y = (x^2 - 5x + 4)^3$
19. $y = \sqrt{1+e^x}$
20. $y = \frac{1}{\sqrt[3]{x - \ln(x)}}$

For the two year bond P = $\frac{5}{1+R} + \frac{5}{(1+R)^2} + \frac{5}{(1+R)^3} + \frac{105}{(1+R)^4}$

21. Find $\frac{dP}{dR}$,the dollar duration.

22. Find $\frac{d^2P}{dR^2}$, the dollar convexity.

23. A particle travels on a line such that its position ,s , is a function ,f ,of time ,t ,given by $s = \frac{1}{3}t^3 - 2t^2 + 3t$; $0 \le t$



a. When is $V_t = 0$? Where is the particle?
b. When is $a_t = 0$? Where is the paricle?

24. If z = (fog)(x) = f(g(x)), the composite of f with g, then

$$\frac{dz}{dx} = f'(g(x)).g'(x)$$

if

x	0	1	2	3	4
g(x)	3	4	3	2	0
$g'(x)$	1	2	4	7	9
f(x)	0	8	1	-3	-5
$f'(x)$	2	4	6	5	1

then find $\frac{dz}{dx}(2) = (fog)'(2)$

25. Determine which statements are true (T) or false (F).

a. $V_t = \left(a_t\right)'$
b. $\left(f.g\right)' = f'.g'$
c. $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
d. acceleration is the 2nd derivative of position with respect to time.
e. $\left(x^{m/n}\right)' = x^{m-n}$
f. The derivative of a quadratic function is a linear function.
g. $\left(f/g\right)' = f'/g'$