Lesson 4
Linear Algebra

A family of vectors is **linearly independent** if none of them can be written as a linear combination of finitely many other vectors in the collection. Consider ***m*** linearly independent equations in ***n*** unknowns:

$$a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n = \sum_{j=1}^{n} a_{i,j}x_j = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n = \sum_{j=1}^{n} a_{i,j}x_j = b_2$$

$$a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n = \sum_{j=1}^{n} a_{i,j}x_j = b_m$$

There are 0, 1 (called a nonsingular system), or infinitely many solutions.

Definitions:

a. If $m = n$, then we call the matrix a **square matrix**.
b. Matrix A = matrix B if and only if they each have the same number of rows and the same number of columns and each element of A equals each element of B.
c. The **zero matrix** is where every element is zero.

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

d. A matrix consisting of one row and several columns is called a **row matrix** or a **row vector.**

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \end{bmatrix}$$

e. A matrix consisting of one column and several rows is called a **column matrix** or a **column vector**.

$$\begin{bmatrix} a_{1.1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}$$

f. If $m = n$, then the **identity matrix** is the matrix with 1s along main diagonal and zeroes everwhere else.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

g. If A has m rows and n columns then A is an $m \times n$ matrix.

A. Row Reduction

Betty and Bob find treasure at an oasis. They will take home gold, water and fuel. Each container of gold weighs 10 lbs, takes 6 hrs to pack and occupies 2 cu.ft. Each container of water weighs 2lbs, takes 4hrs to pack, and occupies 4 cu.ft. Each container of fuel weighs 2lbs, takes 3hrs to pack, and occupies 3 cu.ft. Their vehicle must take 100 lbs, they must use 120 hrs and 100 cu.ft. How many containers of gold, water, and fuel must they take?

Let x_1 = the number of containers of gold
x_2 = the number of containers of water
x_3 = the number of containers of fuel

Consider the following system of linear equations:

$$\begin{aligned} 10x_1 + 2x_2 + 2x_3 &= 100 \\ 6x_1 + 4x_2 + 3x_3 &= 120 \\ 2x_1 + 4x_2 + 3x_3 &= 100 \end{aligned}$$

We can replace any equation in the system by a multiple of itself or a multiple of itself ± a multiple of another equation in the system. This is called **row reduction.**

For example, the equation

$$10x_1 + 2x_2 + 2x_3 = 100$$

is given as true. Therefore, by multiplying both sides by 2:

$$20x_1 + 4x_2 + 4x_3 = 200$$

or, dividing both sides by 10:

$$x_1 + .2x_2 + .2x_3 = 10$$

are also true equations. Also, we can subtract one true equation from another so that:

$$\begin{aligned} 10x_1 + 2x_2 + 2x_3 &= 100 \\ -(\ 6x_1 + 4x_2 + 3x_3 &= 120\ ) \\ \hline 4x_1 - 2x_2 - x_3 &= -20 \end{aligned}$$

is also a true equation. Let's begin using these rules to solve the system of linear equations.

$$\left[\begin{array}{ccc|c} 10 & 2 & 2 & 100 \\ 6 & 4 & 3 & 120 \\ 2 & 4 & 3 & 100 \end{array}\right] \mapsto R_1 \to \frac{R_1}{10} \quad \left[\begin{array}{ccc|c} 1 & 1/5 & 1/5 & 10 \\ 6 & 4 & 3 & 120 \\ 2 & 4 & 3 & 100 \end{array}\right] \mapsto R_2 \to R_2 - 6R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1/5 & 1/5 & 10 \\ 0 & 14/5 & 9/5 & 60 \\ 2 & 4 & 3 & 100 \end{array}\right] \mapsto R_3 \to R_3 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 1/5 & 1/5 & 10 \\ 0 & 14/5 & 9/5 & 60 \\ 0 & 18/5 & 13/5 & 80 \end{array}\right] \mapsto R_2 \to R_2 \frac{5}{14}$$

$$\left[\begin{array}{ccc|c} 1 & 1/5 & 1/5 & 10 \\ 0 & 1 & 9/14 & 150/7 \\ 0 & 18/5 & 13/5 & 80 \end{array}\right] \mapsto R_3 \to R_3 - \frac{18}{5} R_2 \quad \left[\begin{array}{ccc|c} 1 & 1/5 & 1/5 & 10 \\ 0 & 1 & 9/14 & 150/7 \\ 0 & 0 & 2/7 & 20/7 \end{array}\right]$$

Now,

$$\frac{2}{7} x_3 = \frac{20}{7}$$

So, $x_3 = 10$, and therefore,

$$x_2 + \frac{9}{14} \cdot 10 = \frac{150}{7}$$

So, $x_2 = 15$, and therefore,

$$x_1 + \frac{1}{5} \cdot 15 + \frac{1}{5} \cdot 10 = 10$$

So, $x_1 = 5$.

B. Cramer's Rule

Let's start again with the previous system of linear equations, augmenting the matrix of coefficients to get:

$$\begin{array}{ccc|ccc} 10 & 2 & 2 & 10 & 2 & 2 \\ 6 & 4 & 3 & 6 & 4 & 3 \\ 2 & 4 & 3 & 2 & 4 & 3 \end{array}$$

Taking the down diagonals and the up diagonals, we get:

10 × 4 × 3 =	120	2 × 4 × 2 =	16
2 × 3 × 2 =	12	4 × 3 × 10 =	120
2 × 6 × 4 =	48	3 × 6 × 2 =	36
Sum:	180	Sum:	172

Now, 180 – 172 = 8, the matrix determinant, D, or |A|. Start again, augmenting the matrix of coefficients as shown:

$$\begin{array}{ccc|ccc} 100 & 2 & 2 & 100 & 2 & 2 \\ 120 & 4 & 3 & 120 & 4 & 3 \\ 100 & 4 & 3 & 100 & 4 & 3 \end{array}$$

Taking the down diagonals and the up diagonals, we get:

100 × 4 × 3 =	1200	100 × 4 × 2 =	800
2 × 3 × 100 =	600	4 × 3 × 100 =	1200
2 × 120 × 4 =	960	3 × 120 × 2 =	720
Sum:	2760	Sum:	2720

Now, 2760 – 2720 = 40, Dx_1. Start again, augmenting the matrix of coefficients as shown:

$$\begin{array}{ccc|ccc} 10 & 100 & 2 & 10 & 100 & 2 \\ 6 & 120 & 3 & 6 & 120 & 3 \\ 2 & 100 & 3 & 2 & 100 & 3 \end{array}$$

Taking the down diagonals and the up diagonals, we get:

10 × 120 × 3 =	3600	2 × 120 × 2 =	480
100 × 3 × 2 =	600	100 × 3 × 10 =	3000
2 × 6 × 100 =	1200	3 × 6 × 100 =	1800
Sum:	5400	Sum:	5280

Now, 5400 – 5280 = 120, Dx_2. Start again, augmenting the matrix of coefficients as shown:

$$\begin{array}{ccc|ccc} 10 & 2 & 100 & 10 & 2 & 100 \\ 6 & 4 & 120 & 6 & 4 & 120 \\ 2 & 4 & 100 & 2 & 4 & 100 \end{array}$$

Taking the down diagonals and the up diagonals, we get:

10 × 4 × 100 =	4000	2 × 4 × 100 =	800
2 × 120 × 2 =	400	4 × 120 × 10 =	4800
100 × 6 × 4 =	2400	100 × 6 × 2 =	1200
Sum:	6880	Sum:	6800

Now, 6880 – 6800 = 80, Dx_3. Using the values we have found we can calculate x_1, x_2, and x_3.

$$x_1 = Dx_1 / D = 40 / 8 = 5$$
$$x_2 = Dx_2 / D = 120 / 8 = 15$$
$$x_3 = Dx_3 / D = 80 / 8 = 10$$

C. Matrix Math

An *m x n* matrix, that is with dimensions *m* by *n*, is a array of *m* rows and *n* columns of numbers, called elements.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,n} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

If $m = n$ then we call the matrix a square matrix. A = B, if and only if they each have the same number of rows and the same number of columns and every element of A is equal to every element of B; that is $a_{ij} = b_{ij}$ for all i,j. A matrix is said to equal 0, if and only if every element is equal to zero. That is, if A = 0, then $a_{ij} = 0$.

$$A = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A matrix that consists of one row and several columns is called a row matrix or a row vector.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \end{bmatrix}$$

A matrix that consists of one column and several rows is called a column matrix or a column vector.

$$\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \dots \\ a_{m,1} \end{bmatrix}$$

If $m = n$ then I, the identity matrix, has ones along the main diagonal.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

D. Addition and Subtraction

To add two matrices or to subtract one matrix from another, the two must have the same dimensions and:

$$\lfloor a_{i,j} \rfloor \pm \lfloor b_{i,j} \rfloor = \lfloor a_{i,j} \pm b_{i,j} \rfloor$$

For example,

$$\begin{bmatrix} 4 & 3 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 11 & 9 \end{bmatrix}$$

E. Scalar Multiplication

To multiply a matrix by a scalar, that is a one by one matrix, multiply each element in the matrix by the scalar.

$$c\lfloor a_{i,j} \rfloor = \lfloor c \cdot a_{i,j} \rfloor$$

For example,

$$3 \cdot \begin{bmatrix} 1 & 6 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 18 \\ 12 & 21 \end{bmatrix}$$

F. Matrix Multiplication

In order to multiply two matrices together, say A times B, the number of columns in A must equal the number of rows in B. The solution matrix will have dimensions equal to the number of rows in A and the number of columns in B. Then, take the i^{th} row of A and multiply by the j^{th} column of B. For example,

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 2\cdot 4 + 3\cdot 2 & 2\cdot 8 + 3\cdot 6 \\ 1\cdot 4 + 7\cdot 2 & 1\cdot 8 + 7\cdot 6 \end{bmatrix} = \begin{bmatrix} 14 & 34 \\ 18 & 50 \end{bmatrix}$$

Note that matrix multiplication is non commutative. So, BA ≠ AB.

G. Matrix Inversion

Consider a square matrix, A. If there exists a square matrix ,B, such that AB = BA = I, then B is called the inverse of A. That is, B = A^{-1}. Also then, A is said to be invertible or nonsingular. Matrix A is nonsingular if and only if D(A) ≠ 0. Consider a system of linear equations such that AX = B. If A is nonsingular, then $A^{-1}AX = IA^{-1}B$. Then, X = $A^{-1}B$ is the solution. To find A^{-1}, where:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

To find A^{-1}, augment A to get [A | I].

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Then, apply row reduction to find [I | A^{-1}].

$$A^{-1} = \begin{bmatrix} 3.5 & -1.5 \\ -2 & 1 \end{bmatrix}$$

Given that AX = B, the system can be solved by computing $A^{-1}B$.

H. Eigenvalues and Eigenvectors

An eigenvector of a given linear transformation is a non-zero vector which is multiplied by a constant called the eigenvalue as a result of that transformation. An important tool for describing eigenvalues of square matrices is the characteristic polynomials, where λ is an eigenvalue of A is equivalent to (A – λI) v = 0 has a non-zero solution v, an eigenvector, and so it is equivalent to the matrix determinant, det(A – λI), being zero. It follows that we can compute all the eigenvalues of a matrix A by solving the equation $p_A(\lambda) = 0$. If A is an n-by-n matrix, then p_A has degree n and A can therefore have at most n eigenvalues.

Project 4

1. Use the Row Reduction Method to solve

$$x - y - z = 1$$
$$2x - 3y + z = 10$$
$$x + y - 2z = 0$$

2. Use the Row Reduction Method to solve

$$x - y + z = 6$$
$$3x - y + 11z = 6$$
$$2x + y + 4z = 8$$

3. Use the Row Reduction Method to solve

$$x + 5y + 3z = 7$$
$$2x + 11y - 4z = 6$$

4. Use Cramer's Rule (determinants) to solve

$$x + y + z = 6$$
$$3x + 2y - z = 4$$
$$3x + y + 2z = 11$$

5. Solve the <u>non linear</u> system

$$x^2 + y^2 = 4$$
$$-2x + y = -1$$

Interpret your results geometrically.

6. Find the dimensions of the following matrices ("mxn")

a. $\begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

b. $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

c. $\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

d. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$

e. $\begin{pmatrix} 5 & -7 & 2 & 5 \\ 1/3 & -1/2 & 6 & 4 \end{pmatrix}$

f. (2)

7. a. Find $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 3 \end{pmatrix}$

b. Find $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{pmatrix}^2$ i.e the product of the matrix with itself. Any Comments?

8. Let A = $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. If A^2 +A = 0 (the zero matrix) then find a and b.

9. Use the inverse matrix method to solve

$$x + 2y + 3z = 3$$
$$2x + 5y + 7z = 6$$
$$3x + 7y + 8z = 5$$

10. (Portfolio Problem) A portfolio of 3 stocks have table

	A	B	C	Portfolio
Proportion	X	Y	Z	1
Return	.20	.40	.50	.45
Risk(beta)	1	2	5	2.80

Use the inverse matrix method to solve the system of 3 equations and 3 unknowns that is a result of this information. What are your proportions?