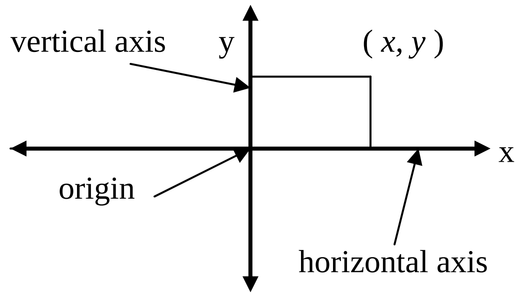
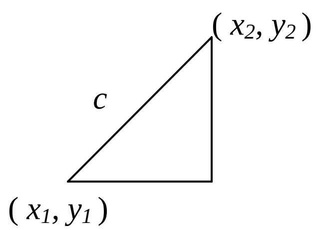
Lesson 3

Functions and Analytic Geometry

A point in the plane is located by an ordered pair of numbers (x, y) as per:



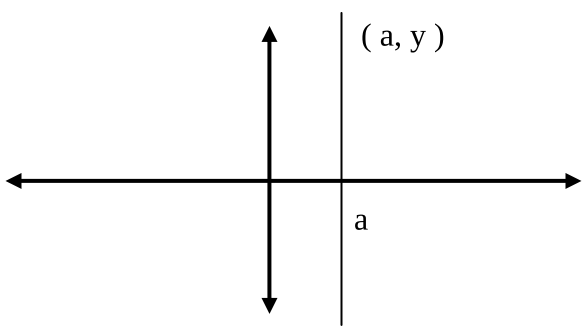
This is called a rectangular or Cartesian coordinate system. Based on the Pythagorean theorem the distance c between two points is given by



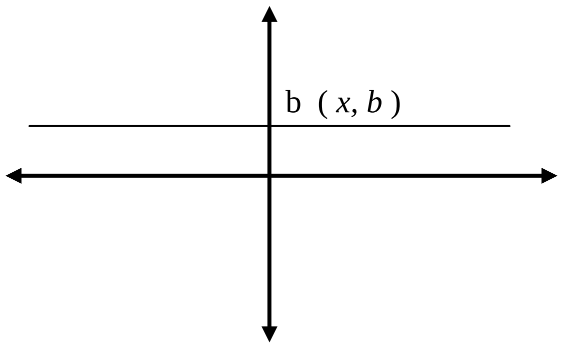
$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Ex. 1: Find the distance between (-3, 2) and (2, 14).

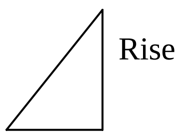
Vertical lines are fully described by an equation as $x = a$:



Horizontal lines are fully described by an equation as per $y = b$:



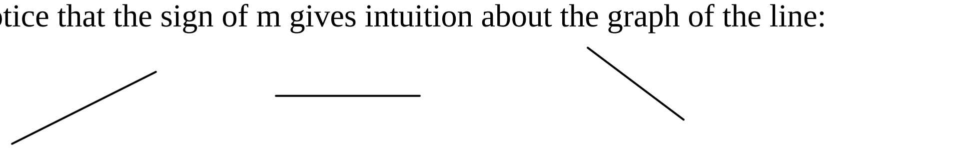
The slope of a non-vertical line can be found. 2 points define a line with a rise—the change in *y*—and a run—the change in *x*. The rise divided by the run is the slope.



$\frac{rise}{run}$ = constant called the slope *m*.

Rule: $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Notice that the sign of m gives intuition about the graph of the line:



ascending ($0 < m$)   horizontal ($m = 0$)   descending ($m < 0$)   vertical (*m* undefined)

The **equation of a line** is formed by considering the point (0, *d*) on the line and the slope *m*. *d* is called the *y*-axis intercept.

Rule: Slope-Intercept Form: $y = mx + d$

Ex. 2: Given that (0, 1) and (4, 3) define a line, what is the equation of the line?

$$m = \frac{3-1}{4-0} = \frac{1}{2}$$

$$y = \frac{1}{2}x + 1$$

Rule: Slope-Point Form: $y - y_1 = m (x - x_1)$

More often we start with the slope-point form equation of a line, then work back to the slop-intercept form.

Ex. 3: Find the equation of the line passing through (1, 2) and (3, 7).

$$m = \frac{7-2}{3-1} = \frac{5}{2}$$

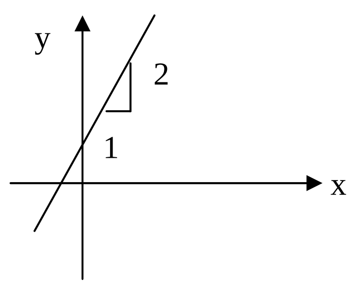
$$y - 2 = \frac{5}{2}(x-1)$$

$$y - 2 = \frac{5}{2}x - \frac{5}{2}$$

$$y = \frac{5}{2}x - \frac{1}{2}$$

We can examine functions using equations, graphs, and tables.

$y = 2x + 1$

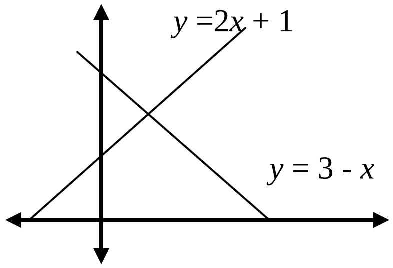


x	-1	0	1	2	3
y	-1	1	3	5	7

Rule: *y* vs. *x* is *linear* if and only if equal changes in *x* result in equal changes in *y*. The ratio of changes is the slope.

Notice that if two lines have the same slope, they are parallel $m_1 = m_2$. If they have different slopes, they are intersecting $m_1 \neq m_2$.

Ex. 4:

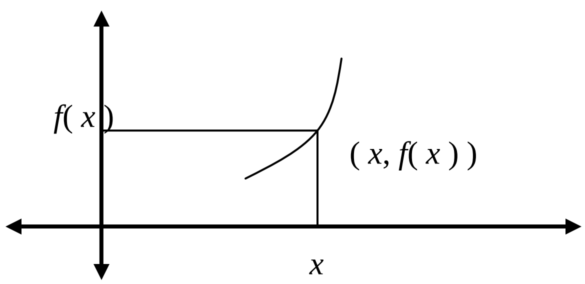


How do we find the point of intersection? Set the two equations equal to each other and solve for x.

$$2x + 1 = 3 - x$$
$$3x = 2$$
$$x = 2 / 3$$
$$y = 3 - 2 / 3 = 7 / 3$$

So, the point of intersection is (2 / 3, 7 / 3).

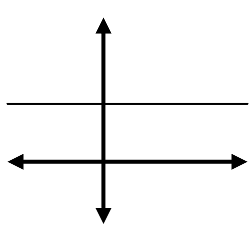
A **function**, usually denoted by *f* but also by *g* and *h* at times, is a rule that assigns to each input *x* a unique output, *f*(*x*), called the value of the function at *x*. The collection of all points in the plane (*x*, *f*(*x*)) is called the graph of *f*.



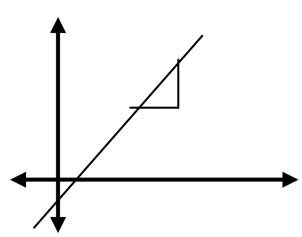
We study function using tables, graphs, and sometimes formulae, if they exist.

Examples:

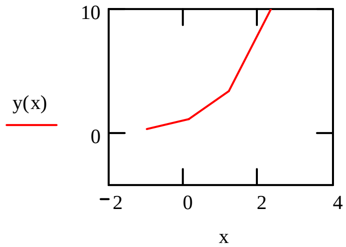
y = c



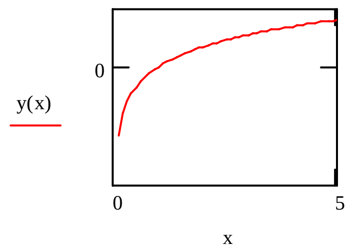
y = mx+d



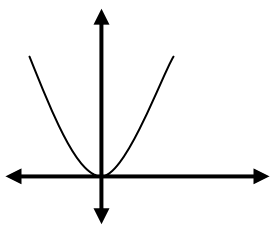
$y(x) := e^x$



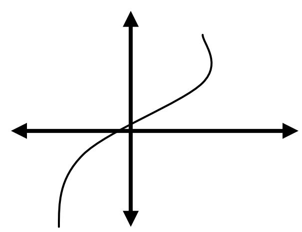
$y(x) := \ln(x)$



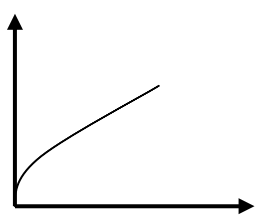
$y=x^2$



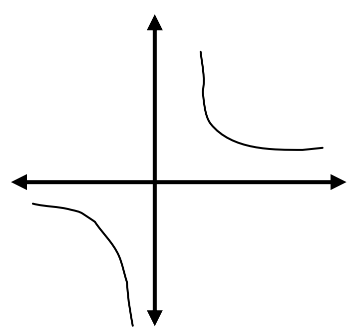
$y=x^3$



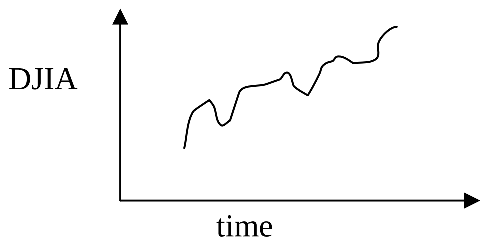
$y=\sqrt{x}$



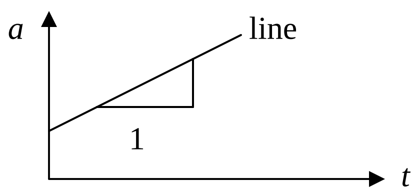
$y=\frac{1}{x}$



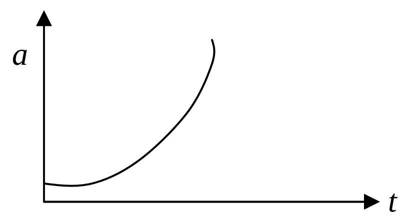
For markets, price vs. time is a function, called a time series. Notice that there is no equation for this function!



For simple interest, if p and r are constant, then a vs. t is a function for $a = p(1 + rt)$



For compound interest, if p and r are contstant, then a vs. t is a function for $a = p(1 + r)^t$



A 2 year bond price p is a function of yield r, where c is coupon.

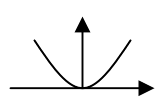
$$P = \frac{C}{1+R} + \frac{C}{(1+R)^2} + \frac{C}{(1+R)^3} + \frac{C+100}{(1+R)^4}$$

Definition: A **quadratic function** is a function that can be written in the form.

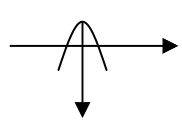
$$y = ax^2 + bx + c$$

a ,b, c are numbers called the coefficients and $a \neq 0$.

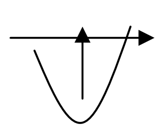
$y = x^2$

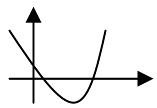


$y = 4 - x^2$



$y = 2x^2 - x - 3$



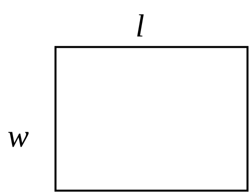
$y = (x-2)(x-3)$ 

Rule: The graph of a quadratic function is a parabola:

1. if $0 < a$, then the graph is concave up. i.e smiling .
2. if $a < 0$, then the graph is concave down. i.e frowning .
3. The graph cuts the y-axis at (0, c)
4. The vertex is $\left(-\dfrac{b}{2a}, c - \dfrac{b^2}{4a}\right)$
5. The graph cuts the *x*-axis at *x* such that $ax^2 + bx + c = 0$. These are the roots or x-intercepts of the function.

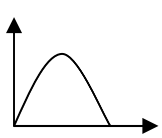
Word Problem

Betty and Bob have 100ft of lumber to make a rectangular pen. What should the dimensions be to maximize the area? What is the maximum area?



$2w + 2l = 100$
$A = wl$
$2l = 100 - 2w$
$l = 50 - w$
$A = w(50 - w) = -w^2 + 50w$

Clearly, $0 < w < 50$
$A \quad (w_{max}, A_{max})$



Now A is a quadratic function of w, so a vs. w is:
0 50
Now $a = -1$ so $a < 0$, hence the vertex is a maximum point.

$$w_{max} = -\frac{b}{2a} = -\frac{50}{2(-1)} = 25$$

$$A_{max} = c - \frac{b^2}{4a} = 0 - \frac{(50)^2}{4(-1)} = 625$$

Clearly the figure is a square; 25 x 25 with maximum area of 625 sq ft.

Rule: To find the points of intersection of the graphs of 2 functions set the formulae equal to one another and solve.

Example: Find the points of intersection between $y = x^2 - 6x + 11$ and $y = 2 - x^2$

Consider the function $y = f(x) = x^2 + 1$.

Clearly, $f(5) = 5^2 + 1 = 26$

and $f(-3) = (-3)^2 + 1 = 10$.

What is $f(u + 2)$?

$$\begin{aligned} f(u+2) &= (u+2)^2 + 1 \\ &= u^2 + 2u + 4 + 1 \\ &= u^2 + 2u + 5 \end{aligned}$$

Rule: To evaluate $y = f(x)$ at an algebraic expression replace the independent variable, x, in the original defining equation by the algebraic expression and algebraically simplify, if possible.

Ex. 5: If $f(x) = x^2$, then $\dfrac{f(x+h) - f(x)}{h}$ is found as per:

$f(x) = x^2$

$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$

$f(x + h) - f(x) = 2xh + h^2$

$$\frac{f(x+h) - f(x)}{h} = \frac{h(2x+h)}{h} = 2x + h$$

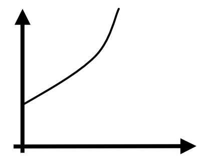
Ex. 6: If $f(x) = 3x + 5$, then find $\dfrac{f(x+h) - f(x)}{h} =$

A function that can be written as per $y = a \cdot b^{kx}$, where $0 < a$ and $0 < k$, is called a **exponential growth or decay function**

Examples:

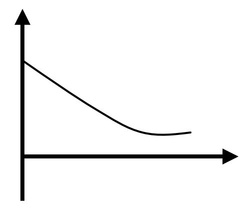
$y = 5 \cdot 2^x$

x	0	1	2	3	4
y	5	10	20	40	80



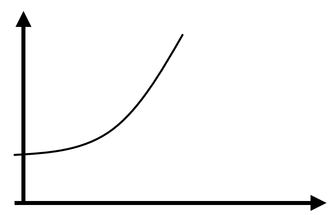
$y = 60 \cdot (1/3)^x$

x	0	1	2	3	4
y	60	20	6.67	2.22	.74



$y = 10 \cdot 3^{5x}$

x	0	2	4	6	8
y	10	30	90	270	810



1.

x	0	1	2	3	4
y	5	10	20	40	80

2.

1 1 1 1

x	0	1	2	3	4
y	60	20	6.67	2.22	.74

1/3 1/3 1/3 1/3

3.

2 2 2 2

x	0	2	4	6	8
y	10	30	90	270	810

3 3 3 3

Rule: If $1 < b$, then exponential growth.
If $0 < b < 1$, then exponential decay.
Observe $0 < y$ that is no x intercepts.

Rule: y vs. x is exponential growth or decay if and only if equal changes in x produce changes in y such that consecutive quotients are constant. This constant is the growth or decay multiple b.

Compound interest $a = p(1 + r)^t$ is exponential growth or decay, for clearly $b = 1 + r$.

War of the Worlds

Earth is invaded by monsters from Mars! 50 monsters are initially present and they double every day. Mother Nature produces 1 bacterium and it triples every day. If the monsters can survive 10 days then the earth is doomed. If the bacteria equal the number of monsters then the monsters are doomed. Is the earth saved or doomed?

Let m = the number of monsters and b = the number of bacteria. Let t = the number of days. Clearly,

$$m = 50 \cdot 2^t \quad \text{and} \quad b = 1 \cdot 3^t$$

We must solve to find when these two exponential growth functions are equal. Now,

$$50 \cdot 2^t = 1 \cdot 3^t$$

$$50 = 3^t / 2^t = (3/2)^t = (1.5)^t$$

So,

$$\ln(50) = \ln(1.5)^t = t \cdot \ln(1.5)$$
$$t = \ln(50) / \ln(1.5) = 1.69897 / .17609$$

Thus, $t = 9.648$ and the earth is saved!

We conclude by making the following comments concerning functions in general. The concept involves the relationship between 2 sets of numbers. Therefore any y vs. x as per

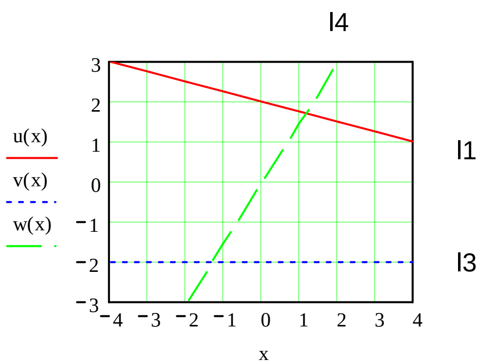
x	x_1	x_2	x_3	…
y	y_1	y_2	y_3	…

The equation may not be known, but the graph can be found by plotting points or using a graphing calculator or computer software program. A function is a particular tabular relationship which allows us to efficiently study financial relationships!

Project 4

1. Find the distance between the following pair of points
   a. (4,-2) and (-1,7)
   b. (2,5) and (3,1)
   c. (0,0) and (3,4)
   d. (-4,3)and (-4,-5)

2. Find the equation of the lines as per graph , state your answer in slope-intercept form



l2 vertical cuts x-axis at -3

3. Find the point of intersection of these lines from problem2
   a. l_1 and l_4 b. l_3 and l_4 c. l_2 and l_3

4. Find the equation of the line passing through (-1,3) and (4,-9)
5. Find the equation of the line passing through (2,-1)and which is parallel to 4x-5y = 10.
6. Find the equation of the line that cuts the x-axis at –2 and the y axis at –5.
7. If

x	-5	a	5	10
y	b	3	6	8

   is linear then find a and b.

8. Determine where the line –8x+2y = 12 cuts the x-axis and the y-axis . Now draw the graph of this line. What is its slope?
9. Let f(x) = $\sqrt{25-x^2}$. Find f(4), f(6) . What is the domain of f , the set of all possible inputs ?

10. Let y = $\begin{cases} 2 \;\; if \;\; x \le -1 \\ x^3 \; if \; -1 < x < 1 \\ \sqrt{x} \; if \; 1 \le x \end{cases}$

Draw the graph of y vs x , neatly , on a rectangular coordinate system.

11. Let $y = 6\text{-}x\text{-}x^2 = f(x)$

a. Describe the graph of f.
b. Find the y – intercept
c. Find the maximum or minimum point (which is it?)
d. Find the x- intercepts , if they exist.
e. Neatly draw the graph of f on a rectangular coordinate system.

12. Algebraically find , if they exist , the points of intersection of

$y = 3x^2\text{-}6x\text{-}13$ and $y = x^2\text{-}7x+2$

13. Algebraically find , if they exist , the points of intersection of

$y = x^2\text{-}x\text{-}9$ and $y = 2x+1$

14. Let f(x) = $\dfrac{2}{x-1} - 4$. Algebraically find where the graph of f cuts the x-axis and where it cuts the y-axis.

15. If

X	0	1	2	3	4
Y	12	30	75	187.5	468.75

then find a formula for y vs x.

16. If

X	0	4	8	12	16
Y	800	600	450	337.5	253.13

then find formula for y vs x.

17. Ann's Ant Farm starts with 8 ants. The ants triple every week.

a. Express the amount ,A, as a function of the time ,t , in weeks.
b. If 1000ants weigh 1 pound and ants sell for $5 per pound then what is the farm worth in 13 weeks?

18. The Continuous Compound Interest Formula is

$A = Pe^{RT}$

a. If Winona triples her money every 5 years then find her continuous rate of return, R.
b. Generalize your work for every T years . What is R ?
c. Further generalize your work for N tuple and T years. What is R?

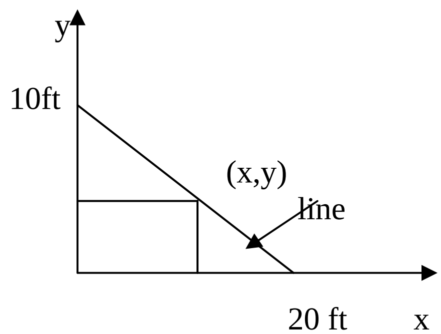
19. Leonardo has an account whose value over time is as per:

T	1	3	5	7
A	15	18	21	24

T in years , A in thousands of dollars

a. Find A vs T if the pattern continues for all time .
b. What was his initial investment ?
c. What is his simple per annum rate of return?

20. The NUTEX exchange is constructing air conditioning duct under its roof as per the cross sectional diagram:



The duct has rectangular cross section inscribed as per figure. Determine the dimensions of the rectangle that maximize the area. What is this maximum area ?

21. For the 2 year bond: P= $\frac{5}{1+R}+\frac{5}{(1+R)^2}+\frac{5}{(1+R)^3}+\frac{105}{(1+R)^4}$

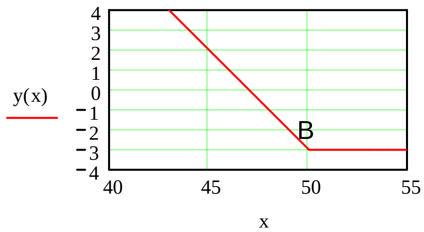
a. Find P if R = .03
b. Find P if R = .05
c. Find P if R = .07
d. As R increases does P increase or decrease ?

22. Let f(x) = $x-x^2$. Find $\frac{f(x+h)-f(x)}{h}$ and simplify as much as possible.

23. (Put Option P&L) The holder or buyer of a put option has the right to sell the stock at the exercise price on or before expiration : For the P&L versus stock as

Consider an Oct 50 put purchased at $3 per share.

a. Find the formula for y vs x.
b. Algebraically compute the breakeven point .
c. If x = 40 then find y. If x = 55 then find y.



B is breakeven

24. Sigourney purchases 1 share of stock at $50.

Let X = the price of the stock 3 months from now

Let Y = the profit or loss she would have if she sells at this price.

a. Make a table of Y vs X for X = 40,45,50,55,60.
b. Make a graph of YvsX. Describe your graph.
c. Find a formula for Y vs X . What type of function is this?

25. Here is a potpourri of problems

a. Find the x and y intercepts of y = $x + \frac{1}{x}$.
b. Carla proposes the rule: The graph of –f = -1.f is the graph of f rotated about the x-axis ,that is –f is the mirror image of f. Comment on this rule with examples.
c. If f(x) = 1 then find f(x+h).
d. Leonardo says that the net asset value (NAV) of a mutual fund is a function of time .Do you agree or disagree . Explain.
e. Define the absolute value of x, denoted $|x|$ by

$$f(x) = \begin{cases} -x \ if \ x < 0 \\ 0 \ if \ x = 0 \\ x \ if \ 0 < x \end{cases}$$

Draw the graph of f. Find f(-7) . Find f(3)

f. Let z = f(y) and y= g(x) be two functions . The composite function ,fog is defined by

Z = (fog) (x) = f(g(x)),

that is f evaluated at g(x).

Let Z = $2y^2+1$ = f(y) and y = 3x-5 = g(x)

Find (fog)(x), the composite of f with g.