Chapter 2

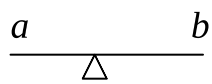
Equations

I. Equations

An **equation** is a statement of the form $a = b$. where a and b are algebraic expressions. Examples are:

$$2 = 6 / 3 \qquad x + 10 = 15 \qquad \frac{2}{x} + 6 = 14 \qquad a = p(1 + rt)$$

An equation can be thought of as a scale in balance.



To remain in balance or true, what you do to one side of the equation, you must do to other side. So,

If $a = b$, then $a \pm c = b \pm c$
If $a = b$, then $ac = bc$
If $a = b$, then $a^c = b^c$
If $a = b$, then $c^a = c^b$
If $a = b$, then $ln(a) = ln(b)$

For an equation involving one variable, the collection of numbers represented by the variable is called the solution set. Finding this set is called solving the equation. For **one equation and one unknown**, solving entails isolating the unknown variable on one side of the equals sign. The procedure used employs the hierarchy of operations in reverse to get an equivalent equation in which the solution is obvious.

Ex. 1: Solve: $3x + 10 = 15$
$3x + 10 - 10 = 15 - 10$
$3x = 5$
$3x / 3 = 5 / 3$
$x = 5 / 3$

Ex. 2: Solve: $10 = 5x / 6$
$10 \cdot 6 = 5x$
$60 / 5 = 5x / 5$
$x = 12$

Ex. 3: Solve: $3x - 5 = 8x + 10$
$3x - 5 + 5 = 8x + 10 + 5$
$3x = 8x + 15$

$3x - 8x = 15$
$-5x / -5 = 15 / -5$
$x = -3$

Ex. 4: Solve: $3(b + 2) - 1 = 5 - (b - 2)$
$3b + 6 - 1 = 5 - b + 2$
$3b + 5 = 7 - b$
$3b + b + 5 - 5 = 7 - b + b - 5$
$4b = 2$
$b = 4 / 2 = 1 / 2$

Ex. 5: Solve: $\frac{2a}{3}a - 3(a+4) = \frac{a}{2}$

$$6\left(\frac{2}{3}a - 3(a+4)\right) = 6\left(\frac{a}{2}\right)$$

$4a - 18(a + 4) = 3a$
$4a - 18a - 72 = 3a$
$-17a = 72$
$$a = \frac{72}{-17}$$
$$a = -4\tfrac{4}{17}$$

Ex. 6: Solve: $\frac{1}{x+1} = \frac{1}{3x}$
$1 \cdot 3x = 1 \cdot (x + 1)$ Cross multiply!

$3x = x + 1$

$2x = 1$
$x = 1 / 2$

Ex. 7: Solve: $\frac{2}{3x} = \frac{x}{2}$

$4 = 3x^2$ Cross multiply!
$x^2 = 4 / 3$
$x = \sqrt{4/3} = (4/3)^{.5}$

For an equation with several variables, use the rules of algebra to isolate one variable on one side. The value on the other side will not be a number, but an algebraic expression.

II. Simple Interest Formula

$a = p(1 + rt)$

Ex. 8: Solve for *r*: $a / p = 1 + rt$

$$a / p - 1 = rt$$

$$r = \frac{a/p - 1}{t}$$

Ex. 9: Solve for *u*: $y = \frac{u - v}{uv}$

Word Problems:

Ex. 1: Uma has $3 million to manage. She decides to allocate twice as much to stocks as to bonds. She decides to allocate $250,000 to commodities. How much should she allocate to stocks and bonds?

$2b + b + 250 = 3000$

Ex. 2: Winona invests $1000 in stock market. After 9 months her account is worth $1200. What is her simple rate of return per annum?

$1200 = 1000(1 + .75r)$

For **two equations and two unknowns**, solve one of the equations in terms of one of the variables, then substitute that value for the variable in the other equation.

Ex. 3: Tom and Nicole invested $6000, part at 6% per annum simple interest in a C.D., the rest at 8% per annum simple interest in a bond. If the total interest for the year was $394 then how much was invested in each security?

Eq. 1: $c + b = 6000$
Eq. 2: $.06c + .08b = 394$

$$c = 6000 - b$$
$$.06(6000 - b) + .08b = 394$$
$$360 - .06b + .08b = 394$$
$$.02b = 34$$
$$b = 1700$$
$$c = 4300$$

III. Quadratic Equations

A quadratic equation is an equation that can be written as: $ax^2 + bx + c = 0$, where *a*, *b*, *c* are numbers called the coefficients and $a \neq 0$ is called the leading coefficient. This is called the standard form. Examples include:

$x^2 = 100$ $x^2 = -1$ $x^2 - 2x - 35 = 0$ $(x + 9)(x - 8) = 0$

Ex. 1: Solve: $2460 - 18x^2 = 0$

$18x^2 = 2460$

$x^2 = 136.667$

$x = \pm\sqrt{136.667} = \pm 11.69$

Rule: Quadratic Formula

$ax^2 + bx + c = 0$

To solve a quadratic equation in the standard form, use the quadratic formula.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula has 2, 1 or 0 solutions:

1. If $0 < b^2 - 4ac$, then 2 solution.
2. If $0 = b^2 - 4ac$, then 1 solution.
3. If $0 > b^2 - 4ac$, then 0 solutions.

Ex. 1: Solve: $x^2 - 2x - 35 = 0$

a = 1, b = -2, c = -35

Use the quadratic formula:

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-35)}}{2(1)}$$

$$\frac{2 \pm \sqrt{4 + 140}}{2}$$

$$\frac{2 \pm \sqrt{144}}{2}$$

$$\frac{2 \pm 12}{2}$$

$$x_1 = \frac{2 + 12}{2} = 7$$

$$x_2 = \frac{2 - 12}{2} = -5$$

IV. Logarithms

RULES:

1. $\ln(e^r) = r$
2. $e^{\ln(u)} = u$
3. $\ln(a \cdot b) = \ln(a) + \ln(b)$
4. $\ln(a / b) = \ln(a) - \ln(b)$
5. $\ln(a^r) = r \cdot \ln(a)$

Ex. 1: Solve: $1{,}000{,}000 = 3^t$

Ex. 2: Solve: $100 = 1024\,(.5)^t$

Compound Interest

$$a = p(1 + r)^t$$

Ex. 3: Solve for *t*. $\ln(a) = \ln(p) + \ln((1 + r)^t)$

$\ln(a / p) = t \cdot \ln(1 + r)$ See rules 4 and 5.

$$t = \frac{\ln(a/p)}{\ln(1+r)}$$

Note that this solution is ***not*** equal to: $\ln\left(\frac{a/p}{1+r}\right)$

Word Problem

Betty and Bob invest $10,000 at 8% per annum compounded annually. How long will it take them to have an account value of $15,000?

Ex. 1: $2 \cdot 5^n = 3^{2n-1}$

$\ln(2 \cdot 5^n) = \ln(3^{2n-1})$

$\ln(2) + n \cdot \ln(5) = (2n - 1) \cdot \ln(3)$

$.301 + .699n = (2n - 1) (.477)$

$.301 + .699n = .954n - .477$

$.778 = .255n$

$n = .778/.255$

$n = 3.05$

Continuous Compound Interest

$$a = p \cdot e^{rt}$$

Word Problem

Vicki and Dick invest $5000 for 7.5 years, which results in an account balance of $8000. The continuous rate of return, r , is computed as per:

$8000 = 5000 \cdot e^{7.5r}$
$1.6 = e^{7.5r}$
$\ln(1.6) = \ln(e^{7.5r})$
$\ln(1.6) = 7.5r$
$r = \ln(1.6) / 7.5$
$r = .063$

Ex. 2: Solve for a. $\sqrt{1+b^a} = c$

Ex. 3: $\ln(x) = 3$
$\ln(x) = 3$
$e^{\ln(x)} = e^3$
$x = 20.086$

Ex. 4: $\ln(2x + 1) = 1$

Project 2

Solve each of the following equations or solve for the indicated variable:

1. $4x - 7 = 3(8 - 2x)$

2. $\frac{1}{2}(x-6) = \frac{3}{5}x + 1$

3. $\frac{1}{2}\left[1 + \frac{1}{4}(3x-1)\right] = \frac{2x}{3} - \frac{1}{2}$

4. $3[2x + 1 - 2(2x - 1)] + 4 = 2[1 + 2(3 - x)]$

5. $\frac{2x-3}{4x-5} = \frac{2}{5}$

6. $\frac{2x}{x-3} - \frac{5}{x} = 2$

7. $\frac{4}{x} - \frac{3}{x^2} = \frac{5}{2x}$

8. $2x^2 - 7x + 3 = 0$

9. $5x + 2 = 3x^2$

10. $\frac{1}{F} = \frac{1}{A} + \frac{1}{B}; A$

11. $S = \frac{A - RL}{1 - R}; R$

12. A = P + PRT; P

13. $x^4 + x^2 - 12 = 0$; (hint: let $u = x^2$)

14. $(x - 1)(x + 3) = (x + 2)(x - 3) + 1$

15. $3x(2x - 5) = -4x - 3$

16. $A=2\Pi R(R+4)$; R

17. $(2x+1)^2 = 3(x+1)^2$

18. $4x^2=5x$

19. $D = \frac{360}{T}\left(\frac{100-P}{100}\right); P$

20. $x^2=3-2x$

Give a complete solution to each of the following word problems.

21. Uma went to Marshall Field's and spent $5/9$ of her money on a dress, $1/6$ of her money on a hat, $1/12$ of her money on gloves and had \$26.25 left. How much did she come into the store with?

22. Betty and Bob go to dinner. Betty tips at 15% rate and Bob tips at a 25% rate. The bill for both meals was \$ 48 and the server's total tip was \$10.20. Find the cost (before tip) for each and find the amount of the tip that each left.

23. Solve the one year bond equation:

$$103.7721 = \frac{6}{1+R} + \frac{106}{(1+R)^2} \quad ; 0<R$$

as per discussion in your notes. Use the " substitution method".

24. Use the Simple Interest-Rate Of Return , $A = P(1+RT)$ to answer the following:
a. Tom invests \$10,000. After 9 months his account balance is \$12,000
b. Nicole invests \$8000. After 16 months her account balance is \$11,500.
Explain who did better.

Potpourri

25.

a . Carla proposes the Rule: $\frac{A}{B} = 0$ if and only if A = 0 or B=0 . Comment and make suggestions (if necessary)

b. Carla proposes the Rule: $\sqrt{\frac{1}{A}} = \frac{1}{\sqrt{A}}$; 0<A. Comment and make suggestions (if necessary)

c. Find an example of an equation in one unknown ,x , that has an infinite number of solutions.

d. Find an example of quadratic equation with one solution ;other than $x^2 = 0$.

e. What are the solutions of (x-1)(x-2)(x-3)=0?

f. Find an example of an equation involving only the first power of one unknown , x,that has no solutions.