Ghost of College Math

Michael Modica

This book is dedicated to my high school mathematics teacher, Miss Elsie Luhan, who inspired me to learn and teach.

Introduction

In the Universal Studios movie “Ghost of Frankenstein” the villagers are subjected to the horror of a resurrected monster. Many intelligent students are likewise visited by a horrible monster. It is all the forgotten mathematics that they are expected to know upon entering graduate school or entering an in house training program at work.

Not to worry! I have written *Ghost of College Math* to exorcise all those horrible ghosts. It is based on the lecture notes that I used at The Financial Markets And Trading Program of the Illinois Institute of Technology. The text is currently being used at the Center For Law and Financial Markets at IIT.

I would like to thank Pam Reardon and Jack Wing for their encouragement. I would like to thank Melanie Winter for coordinating all the various aspects of getting the text produced. My thanks to Ben Van Vliet for his suggestions and teaching the course at CLFM. I thank Praveen Posani for his super word processing.

The material is written to be used in a ten week class with active participation by the student. There are 5 chapters on pre-calculus and 5 chapters on calculus. The material is presented in a friendly manner. There are lots of examples and problems to be solved by the student in class. There is a problem set at the end of each chapter. The problems tend to be geared for students who will study the financial markets. I recommend a quiz at the beginning of each lecture and a final exam.

The instructor can involve calculators or computer software are their discretion. There is very little need to edit this material for class presentation. It has already been distilled for such a purpose. Of course the student can use all the large wonderful texts for further and more complete treatments.

I hope the student will enjoy *Ghost Of College Math* and find it useful.

Michael Modica
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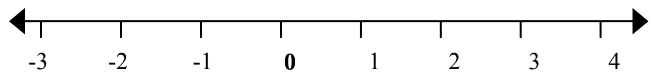
LESSON 1

Algebraic Expressions

I. SIGNED NUMBERS

Signed numbers exist to distinguish between loss and profit, debit and credit, short and long , -100 and 100.



Ex 1: $3 - 7 = -4$
Ex 2: $-7 + 5 = -2$
Ex 3: $9 + (-3) = 9 - 3 = 6$
Ex 4: $-4 + (-2) = -4 - 2 = -6$

RULE: $\mathbf{a + (-b) = a - b}$

II. OPERATIONS

Multiplication and division are done by applying the rule: The product (quotient) of numbers with *like* sign is positive and with *unlike* sign is negative.

Ex. 1: $9 \times 4 = 36$
$9 \times (-4) = -36$
$(-9) \times 4 = -36$
$(-9) \times (-4) = 36$

Ex. 2: $8 / 4 = 2$
$8 / (-4) = -2$
$(-8) / 4 = -2$
$(-8) / (-4) = 2$

Ex. 3: $(-7 + 4) / (5 + (-1)) =$

Ex. 4: $-2 \times (-3 - 5) =$

Ex. 5: $(-3 \times 4) / (2 - 8) =$

RULES:

1. $\mathbf{-a = -1 \times a}$
2. $\mathbf{-(-a) = -1 \times (-a) = a}$

There is a hierarchy of operations in algebra. The mnemonic "My Dear Aunt Sally" may help: multiplication and division first, from left to right; then addition and subtraction, from left to right.

Ex. 1: $6 - 8 + 3 =$

Ex. 2: $7 - 12 / 3 =$

Ex. 3: $(6 + 3 / 4) - (4 + 5 - 5 \times 2) =$

III. INEQUALITIES

Read *a* is less than *b* or *b* is greater than *a*: $a < b$

Ex. 1: $-5 < 2$

$a \le b$ means $a < b$ or $a = b$.
$a < b < c$ means $a < b$ and $b < c$.

Ex. 2: Is $10 \le -20$?
Ex. 3: If $a < 0$ and $0 < b$ is $a < b$?

IV. FRACTIONS

RULES:

1. $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$

2. $\frac{a}{c} x \frac{b}{d} = \frac{axb}{cxd}$

3. $\frac{a / c}{b / d} = \frac{a}{c} x \frac{d}{b} = \frac{axd}{cxb}$

4. $\frac{axb}{cxb} = \frac{a}{c}$

5. $\frac{a}{c} + \frac{b}{d} = \frac{axd}{cxd} + \frac{cxb}{cxd} = \frac{axd + cxb}{cxd}$

6. $\frac{a}{c} = \frac{b}{d}$ if and only if a × d = b × c.

7. $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

Ex. 1: $10\frac{1}{3} = 10 + \frac{1}{3} = \frac{30}{3} + \frac{1}{3} = \frac{31}{3}$

Ex. 2: $10\frac{1}{3} - 8\frac{2}{5} = \frac{31}{3} - \frac{42}{5}$

Ex. 3: $\frac{72}{30} = \frac{2x36}{2x15} = \frac{3x12}{3x5} = \frac{12}{5}$

Ex. 4: 70% of 80 =

V. EXPONENTS

Exponents exist because the product of a number with itself several times often occurs.

Definition of Exponents:

$a^0 = 1$
$a^1 = a$
$a^2 = a \times a$
$a^3 = a \times a \times a$
$a^4 = a \times a \times a \times a$
$a^n = a \times a \times \ldots\ a \times a^{n-1}$

Definition of Negative Exponents:

$a^{-1} = 1 / a$
$a^{-2} = 1 / a^2$
$a^{-3} = 1 / a^3$
$a^{-n} = 1 / a^n$

Ex. 1: $100 \times (1.1)^4 = 100 \times (1.4641) = 146.41$
Ex. 2: $10^{-3} = .001$

RULES:

1. $a^m \times a^n = a^{m+n}$
2. $a^m / a^n = a^{m-n}$
3. $(a^m)^n = a^{m \times n}$
4. $(a \times b)^m = a^m \times b^m$
5. $(a / b)^m = a^m / b^m$

Ex. 1: $(4 u^3 v^4)(7 u^5 v^3) = 28 u^3 u^5 v^4 v^3$
$= 28 u^8 v^7$
Ex. 2: $(a^3 b^5)^4 = a^{12} b^{20}$
Ex. 3: $y^3/y^7 = y^{-4}$

RULE: A factor may cross the fraction sign by changing the *sign* of its exponent from + to – or – to +:

Ex. 4: $\dfrac{a^{-1}}{b} = \dfrac{1}{a^1} \cdot \dfrac{1}{b} = \dfrac{1}{ab}$

Ex. 5: $\dfrac{a}{b^{-1}} = \dfrac{\frac{a}{1}}{\frac{1}{b^1}} = \dfrac{ab}{1} = ab$

Ex. 6: $\dfrac{2^{-1}u^3}{u^{-2}v} =$

Warning: $(a \pm b)^n \neq a^n \pm b^n$

The hierarchy of operations is now extended to: "Egads, My Dear Aunt Sally." Exponents first, then multiplication and division, then addition and subtraction.

Ex. 1: $3 + 4 \cdot 2 - 3^3 / 3 + 4 \cdot 5 =$

Definition of Roots:

$\sqrt{u}$ is a solution of $x^2 = u$; $0 \leq u$. $\sqrt[n]{u}$ is a solution (if it exists) of $x^n = u$.
It is called the *n*th root of *u*.

Ex. 1: $2^4 = 16$ so $2 = \sqrt[4]{16}$
Ex. 2: $10^3 = 1000$ so $10 = \sqrt[3]{1000}$
Ex. 3: $(-5)^3 = -125$ so $-5 = \sqrt[3]{-125}$

RULE: The solutions of $x^n = u$

	u < 0	**0 < u**
***n* even** (2, 4, 6, 8…)	No Solution	2 Solutions $\sqrt[n]{u}$,-$\sqrt[n]{u}$
***n* odd** (3, 5, 7…)	1 Solution $\sqrt[n]{u}$	1 Solution $\sqrt[n]{u}$

So $\sqrt[n]{u}$ doesn't exist if u is negative and n is even (as in the square root).
Note : $\sqrt[n]{0} = 0$ because $0^n = 0$.

RULES:

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b}$

3. $\sqrt[n]{-a} = -\sqrt[n]{a}$ if n is odd.

4. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$

Ex. 1: $\sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2} = 1.5$

Ex. 2: $\sqrt[3]{-8} = -\sqrt[3]{8} = -2$

Ex. 3: $\sqrt[3]{64} =$

Ex. 4: $\sqrt{\sqrt[3]{A}} =$

Ex. 5: $x^3 + 27 = 0$

Ex. 6: $2x^4 - 1250 = 0$

To compute $\sqrt[n]{u}$ compute $u^{1/n}$ using your calculator appropriately.

Ex. 1: $\sqrt[10]{12{,}400} =$

Ex. 2: $\sqrt[5]{-742} =$

Definition of Fractional Exponents:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = \sqrt[n]{a}^{\,m}$$

where $0 < n$ and m may be positive or negative. Fractional exponents obey the laws of exponents.

Ex. 1: $\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{.666}$

Ex. 2: $\sqrt{a^3} =$

Ex. 3: $(\sqrt[4]{16})^5 =$

Ex. 4: $(-27)^{4/3} =$

Ex. 5: $8^{-2/3} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

Ex. 6: $(-125)^{-4/3} =$

Ex. 7: $\sqrt[4]{x^{\frac{3}{2}}.16\sqrt{x}} = \sqrt[4]{x^{\frac{3}{2}}.16x^{\frac{1}{2}}}$

$= \sqrt[4]{16x^2}$

$= 16^{1/4}(x^2)^{1/4}$

$= 2x^{1/2}$

$= 2\sqrt{x}$

Ex. 8: $\sqrt[6]{a^{-3}\sqrt{b^3}} =$

Ex. 9: $x^{1.5} = 100$

Ex. 10: $(x^{1.5})^{\frac{1}{1.5}} = 100^{\frac{1}{1.5}}$

Ex. 11: $x = 100^{.6667}$

RULE: To Solve $x^r = u$, $0 < u$ take the $1 / r$ power of both sides to get $x = u^{1/r}$.

Ex. 12: $a^m + b^n = 1$ Solve for a.

VI. ALGEBRAIC EXPRESSIONS

An algebraic expression consists of numbers, variables (i.e. symbols that represent numbers), operations (e.g. exp, ×, /, +, -) and signs (+, -). Parentheses are used to clarify.

Ex. 1: Simple Interest $a = p(1 + rt)$
Ex. 2: Compound Interest $a = p(1 + r)^t$
Ex. 3: Annuity $s = p((1 + r)^t - 1)$

RULE: Only *like quantities* can be added or subtracted from one another. That is, quantities are like if the variable portions of the algebraic expression are the same.

Ex. 1: $8u^2 - 3uv + 9uv - 19u^2 = 8u^2 - 19u^2 - 3uv + 9uv$
$= -11u^2 + 6uv$

Ex. 2: $-6xy + 7y - 8xy - 12y - 3xy =$

The Distributive Law: $a(b \pm c) = ab \pm ac$

Ex. 1: $-(-2ab) + a(4 - b) = 2ab + 4a - ab$
$= ab + 4a$

Ex. 2: $-x(x - y) + y(3 - 4x) =$

Ex. 3: $\frac{x+1}{3} + \frac{3x}{4} = \frac{4(x+1)}{12} + \frac{3(3x)}{12}$

$= \frac{4x+4+9x}{12}$

$= \frac{13x+4}{12}$

Ex. 4: $\frac{2(x-1)}{3} - \frac{1}{5}x =$

FOIL Rule: (First, Outside, Inside, Last)

$(a+b)(c+d) = ac + ad + bc + bd$

Ex. 1: (3a + 2)(3a – 1) $= 9a^2 - 3a + 6a - 2$
$= 9a^2 + 3a - 2$

Ex. 2: (5u + 9)(2u – 6)=

Factoring by reversing the Foil Rule is: $ab \pm ac = a(b \pm c)$

Ex. 1: $x^2 + 10x = x(x + 10)$

Ex. 2: $ab^2 + ab + a =$

Ex. 3: 3a - 3 = 3a - 3.1 = 3(a – 1)

Ex. 4: $7x^3y - 21x^2y^3 =$

Ex. 5: p + prt =

Ex. 6: $a^2 - 5a + 10 =$

VII. LOGARITHMS

If u > 0, then the logarithm of u, denoted by log(u), is the solution of the equation:

$$10^w = u$$

That is, log(u) is the power to which 10 must be raised to result in *u*. So, $10^w = u$ means $w = \log(u)$. This is called the common or **base 10 logarithm**. Examples of base 10 logarithm are:

Ex. 1: $10^2 = 100$, so log(100) = 2
Ex. 2: $10^6 = 1$ million, so log(1 million) = 6

Ex. 3: $10^0 = 1$, so log(1) = 0
Ex. 4: $10^{-1} = .1$, so log(.1) = -1
Ex. 5: $10^{3.4} = 2511.89$, so log (2511.89) = 3.4
Ex. 6: $10^{-2.5} = .0032$, so log(.0032) = -2.5

To solve an equation in which the unknown is solely in the exponent, take the log of both sides and proceed. This technique often works.

Typically in the social sciences, we use the **natural logarithm** rather than base 10 logarithm. If $u > 0$, then the natural logarithm of u, usually denoted by $\ln(u)$ is the solution of the equation:

$$e^w = u$$

where $e = 2.71828$….. a special number like π. e is called the natural base. You can calculate the natural log on your calculator.

Ex. 1: ln(576) = 6.356
Ex. 2: ln(4780) =
Ex. 3: ln(.085) =

Note that if $0 < u < 1$, then $\ln(u) < 0$. And, if $1 < u$, then $0 < \ln(u)$.

VIII. INVERSE OPERATIONS

Notice that:

$$a + 5 - 5 = a$$
$$a \times 3 / 3 = a$$
$$(a^2)^{1/2} = a$$
$$\ln(e^a) = a$$

IX. SIGMA AND PI NOTATIONS

The ***Sigma notation*** is a handy way to express long sums:

$$a_1 + a_2 + a_3 + \ldots. + a_n$$

as:

$$\sum_{i=1}^{n} a_i$$

Ex. 1: Write $1 + x + x^2 + x^3 + x^4$ using sigma notation.

The ***Pi notation*** is a handy way to express long products:

$$a_1 \times a_2 \times a_3 \ldots \times a_n$$

as:

$$\prod_{i=1}^{n} a_i$$

Ex. 2: Write P = $\dfrac{100}{(1+R_1)(1+R_2)....(1+R_n)}$ using Pi notation.

Project 1

1.a Betty and Bob trade each day of the week with the following results: loss of $150, profit of $130,loss of $40 and a loss of $90. Use signed number calculations to compute their net profit or loss.

b. If Alan is long 500 shares of MCD and then sells 800 shares , is he long or short? Explain mathematically using signed number calculations.

c. Compute : $-9\times2-(-7)+4$

d. Compute : $20-10^2$

e. Carla invests $1000 in Wonder Fund. The fund increases by 18% the first year , increases by 40% the second year , decreases by 65% the third year and increases by 32% in the fourth year . What is the value of her investment after 4 years?

2.a. Betty will make money if at expiration the price of GM is atmost 50 or atleast 55. Use inequality symbols to algebraically describe the prices for which Betty will profit. Now use the number line to describe this set of prices.

b. Bob will sell his position in MCD if MCD has an opening price between 60 and 65(inclusive). Use inequality symbols to algebraically describe the prices for which Bob will sell. Now use the number line to describe this set of prices.

c. Critique the symbolism $8<x<5$. What could the writer be describing on the number line.?

d. Carla proposes a Rule: $a \le a^2$. What comments or suggestions can you make to Carla?

e. Carla proposes a Rule: if $a \le b$ then $ac \le bc$. What comments or suggestions can you make to Carla?

3. Using only fractions and their rules compute , simplify ,or reduce the following :

a $\dfrac{3\tfrac{2}{3} - 4\tfrac{1}{8}}{2\tfrac{1}{6}}$

b. $1\tfrac{1}{2} + 2\tfrac{1}{3} - 4\tfrac{3}{4}$

c. $\dfrac{165}{99}$

Reduce to fraction with smaller components.

d. Linda makes a market as follows : Her bid (what she will pay) is $8\tfrac{19}{64}$; her offer (what she will sell) is 8 ¼. Using fractions, inequalities , the number line ,etc comment on market.

e. . Carla proposes a rule : $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$

What comments or suggestions can you make to Carla?

4.a. Each corn futures contract is based on 5000 bushels of corn . Bob is long 7 contracts. The price of corn trades down 3 ¼ ¢ per bushel. How much does Bob profit or lose? Show your reasoning with mathematics.

b. Each stock option contract is based on 100 shares of stock. Betty is short 4 contracts.Each contract trades down 10¢ per share. How much does Betty profit or lose? Show your reasoning with mathematics.

5.a. Express 1,200,000 in scientific form.
b. Express .0034 in scientific form.
c. Express 2.73×10^{-2} in standard form.
d. Express 3.61×10^{4} in standard form.
e. Express 3 ½ billion in scientific and standard form.

6. Simplify without leaving negative exponents in your final answer.
a. $\left(2x/y^2\right)^4$
b. $5a^{-3}b\left(a^2 + a^4b^{-3}\right)$
c. $\frac{a^{-3}b}{3^{-1}a^2}$
d. $(4uv^{-2}w^3)^{-3}$
e. $\left(-\frac{2}{3}x^{-1}y\right)\left(\frac{3}{5}x^7z^{-1}\right)$

7. Simplify the following expressions:
a . $-2x(5-3y)-(-7x)+yx$
b. $\frac{2x}{3} - \frac{1}{2}(x-5) + \frac{7(x-1)}{4}$
c. $(8u+5)(2u-3)$
d. $(2x-5)(3x^2+x-7)$
e. $\frac{.4(x-1)+.3(2-x)}{.7}$

8. Factor and or simplify the following expressions:
a. $\frac{30x^3-15x}{5x}$
b. $\frac{5}{x^2} - \frac{10}{xy}$

c. $\dfrac{8x^2y + 32xy^4}{xy}$

d. $\dfrac{az^3 - az + 1}{a}$

e. Carla proposes the rule : $\dfrac{ab + c}{a} = b + c$ because like factors cancel! What comments or suggestions can you make to Carla?

9. Here is a potpourri of problems.

a. Express $\left(\dfrac{3}{4}\right)^{-2}$ in fraction and decimal form.

b. Use inequality symbols to describe the set of negative numbers.

c. Use your calculator to compute $(3.45)^6$. Express your final answer to 2 places after the decimal point. Now compute $(3.45)^{60}$. Express your final answer in scientific form, again with 2 places after the decimal point.

d. In solving a problem John argues that

$$-2(4x - 7)^3 = (-8x + 14)^3$$

What comments or suggestions can you make ?

e. Find $(A \pm B)^2$ and $(A \pm B)^3$.

f. Carla proposes the Rule : $0 \le a^2$. What comments can you make ?

g. Use fractions to find $5\tfrac{5}{6} x(-2\tfrac{3}{5})$

h. If commissions are reduced by 10% and then again by 25% then by what percent were the original commissions reduced? Explain with an example . Assume an original payment check of $100.

10. A linear combination of variables such as X,C, and P is an algebraic expression which is the sum and difference of the products of the variable with numbers.

2X-5C+3P, -7X+5C, X+3C-P
are examples of such linear combinations.

The <u>algebraic value</u> , W, of a stock and option position is a linear combination of X,C and P where
X = the price of one roundlot (100 shares) of stock,
C = the value of a call option, and
P = the value of a put option.
So if Bob is long 300 shares and short 8 calls and long 5 puts then

W= 3X-8C+5P.

Note that + is used for long and – is used for short in the algebraic expression.
Write the algebraic value , W, of the following positions:

a. long 500 shares of MCD and short 10 calls.
b. short 800 shares of MCD and long 6 calls and short 9 puts.
c. long 8000 shares of MCD
d. short 40 calls and short 40 puts (short " straddle")
e. long 12000 shares of MCD and long 150 puts .

